New Algorithms for the Perspective-Three-Point Problem

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Abstract In this paper, two approaches are used to solve the Perspective-Three-Point Problem (P3P): the symbolic computation approach and the geometric approach. In the symbolic computation approach, we use Wu-Ritt's zero decomposition algorithm to give a complete triangular decomposition for the P3P equation system. This decomposition provides the first complete analytical solution to the P3P problem. In the geometric approach, we give some pure geometric criteria for the number of real physical solutions. The complete solution classification for two special cases with three and four parameters is also given.

Keywords camera calibration, pose determination, perspective-three-point problem (P3P), analytical solution, geometric criterion, symbolic computation

1 Introduction

The Perspective-n-Point Problem (PnP) is originated from camera calibration[1−3]. It is to determine the position and orientation of the camera with respect to a scene object from n correspondent points. It concerns many important fields, such as computer animation[4], computer vision[3], automation, image analysis and automated cartography[2], photogrammetry[5], robotics[1] and model-based machine vision system[6], etc. Fischler and Bolles[2] summarize the problem as follows:

> "Given the relative spatial locations of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective (C_P), find the lengths of the line segments joining C_P to each of the control points."

The study of the PnP problem mainly consists of two aspects: (1) Design fast and stable algorithms that can be used to find all or some of the solutions of the PnP problem. (2) Give a classification for the solutions of the PnP equation system, i.e., give the conditions under which the system has one, two, three or four solutions. There are many results for the first problem and the second problem is not solved completely.

The P3P problem is the smallest subset of control points that yields a finite number of solutions. The P3P problem was first considered by Grunert, a German mathematician, in 1841. Then Müller outlined and refined Grunert's results in 1925. In 1981, Fischler and Bolles[2] presented the RANSAC algorithm. They have noticed that there are at most four possible solutions to the P3P equation system. Hung *et al.*[7] presented an algorithm for computing the 3D coordinates of perspective center relative to the camera frame. In 1991, Haralick *et al.*[8] reviewed the major direct solutions up to 1991, including six algorithms given by Grunert (1841), Finsterwalder (1903), Merritt (1949), Fischler and Bolles (1981), Linnainmaa *et al.* (1988) and Grafarrend *et al.* (1989), respectively. They presented several

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analysis methods to compare the numerical accuracy and stability of the six algorithms. In 1992, DeMenthon *et al.*[9] showed that by using approximations to the perspective, simpler computational solutions can be obtained.

One of the important research directions on the P3P problem is its multi-solution phenomenon. Fischler and Bolles[2] presented some examples of multi-solution of the P3P problem. In 1986, Wolfe *et al.*[10] pointed out that the six permutations of the three control points combined with four-solution possibility can produce 24 possible camera-triangle configurations consistent with a single perspective view[5,6]. Yuan[5] gave a necessary condition for the existence of the solution for the first time. In 1991, Wolfe *et al.*[6] gave a geometric explanation to this multi-solution phenomenon in the image plane under the assumption of "canonical view".

In 1997, Su *et al.*[4] applied Wu-Ritt's zero decomposition method to the P3P and P4P problems. For the P3P problem, they found the main solution branch and some non-degenerate branches. But a complete decomposition was not given. For the P4P problem, they strictly proved that there are generally no solutions. In [11], they used the Sturm sequence to give improved necessary and sufficient conditions to adjudicate the number of solutions. In 1998, Yang[12] gave solution classifications of the P3P problem under some non-degenerate conditions.

In this paper, we use two approaches to solve the P3P problem: the symbolic computation approach and the geometric approach. In the symbolic computation approach, we apply Wu-Ritt's zero decomposition algorithm[13] to find a complete solution decomposition for the P3P equation system. The decomposition has the following implications. First, it provides a complete analytical solution to the P3P problem. Previous work usually consider the main solutions and omit many special cases. This might cause problems when the given data is from the special case. Second, by expressing all solutions in *triangular form*, it provides a fast and stable way for numerical solution. Third, it provides a clear solution space analysis of the P3P problem and thus provides a good starting point for multiple solution analysis.

In the geometric approach, we consider the three perspective angles separately. Then the locus of the center of perspective point in each case is a toroid and the center of perspective is the intersection of the three toroids. In this way, we give some pure geometric criteria for the number of solutions of the P3P problem. One interesting result is "The P3P problem can have only one solution if all the three angles formed by the three control points are obtuse". Criteria of this kind are given for the first time.

By combining the two approaches, we are able to give the complete solution classification for two special cases of the P3P problem with three and four parameters.

The rest of the paper is organized as follows. In Section 2, we present the algebraic approach. In Section 3, we present the geometric approach. In Section 4, we solve some special cases of the P3P problem completely.

2 The Algebraic Approach

We will use Wu-Ritt's zero decomposition method[13] to tackle the P3P problem. This method may be used to represent the zero set of a polynomial equation system as the union of zero sets of polynomial equations in *triangular form*, that is, equation systems like

$$f_1(u, x_1) = 0, f_2(u, x_1, x_2) = 0, \ldots, f_p(u, x_1, \ldots, x_p) = 0$$

where u could be considered as a set of parameters and x are the variables to be determined. As shown in [13], solutions for an equation system in triangular form are well-determined.

For a polynomial set PS, let $Zero(PS)$ be the set of solutions of the equation system $PS = 0$. Then, we may consider $Zero(PS/G) = Zero(PS) - Zero(G)$ instead of $Zero(PS)$ in order to remove some unnecessary components.

2.1 The Main Component

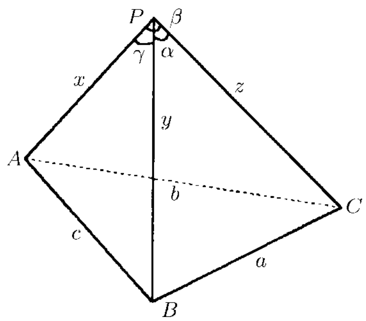

Fig.1. The P3P problem.

Let P be the center of perspective, and A, B, C the control points. Let $|PA| = x$, $|PB| = y$, $|PC| = z$ and $p = \cos\alpha$, $q = \cos\beta$, $r = \cos\gamma$ (Fig.1).

From triangles PBC, PAC, PAB, we obtain the P3P *equation system*:

$$\begin{cases} p_1 = y^2 + z^2 - 2yzp - a^2 = 0 \\ p_2 = z^2 + x^2 - 2zxq - b^2 = 0 \\ p_3 = x^2 + y^2 - 2xyr - c^2 = 0. \end{cases} \tag{1}$$

It is clear that we need to add the following "reality conditions", which are assumed through out the paper.

$$\begin{cases} x > 0, y > 0, z > 0, a > 0, b > 0, c > 0, a + b > c, a + c > b, b + c > a \\ 0 < \alpha, \beta, \gamma < \pi, 0 < \alpha + \beta + \gamma < 2\pi \\ \alpha + \beta > \gamma, \alpha + \gamma > \beta, \gamma + \beta > \alpha. \end{cases} \tag{2}$$

In order to obtain simpler formulas, we introduce the following parameters: $A = \frac{b^2+c^2-a^2}{2}$, $B = \frac{c^2+a^2-b^2}{2}$, and $C = \frac{a^2+b^2-c^2}{2}$, then (1) can be written as follows.

$$\begin{cases} x^2 + yzp - zxq - xyr - A = 0 \\ y^2 + zxq - yzp - xyr - B = 0 \\ z^2 + xyr - yzp - zxq - C = 0. \end{cases} \tag{3}$$

Let $u = p, q, r, A, B, C$ be the parameters, x, y, z the variables to be solved, and

$$G = xyz(p^2 - 1)(q^2 - 1)(r^2 - 1)(Bx_1 + C)(Ax_2 + C)(Ax_3 + B)$$

where x_1, x_2, x_3 are new introduced auxiliary variables. Note that for a new variable x_1, $Bx_1 + C \neq 0$ if and only if $B \neq 0$ or $C \neq 0$. It is clear that $G \neq 0$ under the given "reality conditions" (2).

Applying Wu-Ritt's decomposition method[13] to (3), we obtain a triangular set:

$$\begin{cases} f_1 = I_1^2 x^8 + C_{16}x^6 + C_{14}x^4 + C_{12}x^2 + C_{10} \\ f_2 = xS_2y - C_{20} \\ f_3 = xS_3z - C_{30} \end{cases} \tag{TS_1}$$

where $S_2 = (I_1I_2I_3)^4$, $S_3 = (I_1I_2I_4)^4$, $I_1 = p^2+q^2+r^2-2pqr-1$, $I_2 = Cq(rp-q)+Br(r-pq)$, $I_3 = Cp(rq-p)+Ar(r-pq)$, $I_4 = Bp(rq-p)+Aq(q-rp)$. Detailed coefficients of f_1, f_2, f_3 can be found in [14]. Let $J = xI_1I_2I_3I_4$. From Wu-Ritt's Decomposition Theorem, we have:

$$\mathit{Zero}((3)/G) = \mathit{Zero}(TS_1/J_1) \cup \bigcup_{i=1}^{4} \mathit{Zero}(((3), I_i)/I_1 \ldots I_{i-1}G) \tag{4}$$

where $TS_1 = \{f_1, f_2, f_3\}$, $J_1 = JG$. The first part of (4) is the *main component* for the P3P equation system and the last four zero sets correspond to the special or *degenerate cases*. Notice that the four degenerate components are disjoint to each other. Furthermore, since there is no x, y, z in I_i, the number of solutions in the degenerate case is the maximal number of solutions of the four degenerate components.

The degree eight polynomial equation was also obtained in [4, 8] and from this equation, it is easy to see that at most four solutions can be found for the P3P problem from the main component.

Since $S_2 > 0, S_3 > 0$, $y > 0$ and $z > 0$ are equivalent to $C_{20} > 0$ and $C_{30} > 0$. To find the positive solutions for equation system (1), we need only to solve the following problem: *Determine the positive solutions of* $f_1 = 0$ *under the conditions* $C_{20} > 0$ *and* $C_{30} > 0$. Theoretically, this problem can be solved in many ways. Since f_1 is a quartic equation in x^2, we can solve this equation analytically and use the known results about quartic equations to give the solution criterion. The problem with this simple idea is that the resulted formulas are too complicated to be useful. In principal, this problem can be solved with Tarski-Seidenberg-Collins's quantifier elimination theory[15] or the Sturm-Tarski theory[16]. But, we cannot get a neat solution with all these methods. In [12], a partial solution is given using the complete discriminant method.

2.2 The Degenerate Cases

The degenerate cases are caused by $I_i = 0, i = 1, \ldots, 4$. It is well-known that $I_1 = 0$ if and only if points P, A, B, C are coplanar[11]. Hence $I_1 < 0$ under the reality conditions (2), and we need not consider $Zero(((3), I_1)/G)$.

For $I_2 = 0$, we consider $Zero(((3), I_2)/I_1G)$. Using the Gröbner Basis method[17], we find that the polynomial set $((3), I_2)$ can be decomposed into two branches:

$$Zero(((3), I_2)/I_1G) = \bigcup_{i=1}^{2} Zero(GS_{2i}/I_1G)$$

where

$$GS_{21} = \begin{cases} I_2 \\ (pq - r)z^2 + rC \\ ry - qz \\ (pq - r)(x^2 - 2qxz - A) - Cqp \end{cases}$$

$$GS_{22} = \begin{cases} I_2 \\ rI_1x^2 + Cp(q - rp) + Ar(1 - p^2) \\ rI_1(pq - r)(y^2 - 2rxy) - C_0 \\ (rp - q)z + (pq - r)y + (r^2 + q^2 - 2pqr)x \end{cases}$$

Here $C_0 = C(pr - q)(q^3 - 2pq^2r + qr^2 - q + rp) + Ar(pq - r)(r^2 + q^2 - 2pqr)$. In other words, this degenerate case is reducible. Using Wu-Ritt's algorithm, we have:

$$Zero(GS_{21}/I_1G) = \bigcup_{i=2}^{7} Zero(TS_i/I_1J_iG).$$

$TS_i, i = 2, \ldots, 7$ are given below. J_i is the product of the initials of polynomials (see [13] for definition) in TS_i.

$$\begin{cases} I_2 \\ (rp - q)^2x^4 - 2(rp - q)(T_0 - 2\,qBr^2)x^2 + T_0^2 \\ 2rx(rp - q)y - (pr - q)x^2 + T_0 \\ qz - ry \end{cases} \qquad (TS_2)$$

$$\begin{cases} rp - q \\ B \\ (p^2 - 1)^2x^4 - 2(p^2 - 1)(T_1 - 2Cq^2)x^2 + T_1^2 \\ 2rx(p^2 - 1)y - (p^2 - 1)x^2 + T_1 \\ pz - y \end{cases} \qquad (TS_3)$$

$$\begin{cases} pq - r \\ C \\ (p^2-1)^2x^4 - 2(p^2-1)(T_2 - 2p^2q^2B)x^2 + T_2^2 \\ 2qxp(p^2-1)y - (p^2-1)x^2 + T_2 \\ z - py \end{cases} \qquad (TS_4)$$

$$\begin{cases} q \\ r \\ (p^2-1)x^4 - T_3x^2 + T_4 \\ y^2 + x^2 - A - B \\ pyz + x^2 - A \end{cases} \qquad (TS_5)$$

$$\begin{cases} p \\ Cq^2 - Br^2 \\ x^4 - 2(2Br^2 + A)x^2 + A^2 \\ 2rxy - x^2 + A \\ qz - ry \end{cases} \qquad (TS_6)$$

$$\begin{cases} p \\ q \\ r \\ x^2 - A \\ y^2 - B \\ z^2 - C. \end{cases} \qquad (TS_7)$$

Here $T_0 = A(pr - q) + Bpr$; $T_1 = A(p^2-1) + Cp^2$; $T_2 = A(p^2-1) + Bp^2$; $T_3 = p^2(2A + B + C) - 2A$; $T_4 = p^2(A+C)(A+B) - A^2$. From the computation process, the number of meaningful solutions in this case is the maximal number of solutions of the six components $TS_i, i = 2, \ldots, 7$, which is two.

Similarly, we have

$$Zero(GS_{22}/I_1G) = \bigcup_{i=8}^{12} Zero(TS_i/I_1J_iG)$$

where $TS_8 = GS_{22}, J_8 = rI_1(r - pq)(q - pr)$. Other TS_i are given below.

$$\begin{cases} r \\ C \\ (p^2 - 1 + q^2)x^2 - p^2(B + A) + A \\ (p^2 - 1 + q^2)y^2 - q^2(A + B) + B \\ z - qx - py \end{cases} \qquad (TS_9)$$

$$\begin{cases} r - pq \\ C \\ (q^2 - 1)x^2 + A \\ (q^2 - 1)(y^2 - 2pqxy - B) - q^2A \\ z - qx \end{cases} \qquad (TS_{10})$$

$$\begin{cases} q - pr \\ B \\ (p^2 - q^2)x^2 - p^2A \\ py - qx \\ (p^2 - q^2)(z^2 - 2qxz - C) + q^2A \end{cases} \qquad (TS_{11})$$

$$\begin{cases} p \\ q \\ B \\ (r^2-1)x^2+A \\ y-rx \\ (r^2-1)z^2-r^2(A+C)+C. \end{cases} \qquad (TS_{12})$$

The number of meaningful solutions in this case is two.

From the above decomposition, it is easy to see that there are at most four solutions for case $I_2=0$. The cases for $I_3=0$ and $I_4=0$ can be treated similarly[14]. From Subsections 2.1 and 2.2, we have proved:

Theorem 1. *Under the reality conditions* (2)*, the P3P problem has at most four solutions.*

2.3 Geometric Meaning of the Degenerate Cases

Besides the main component, all the three sub-components given by $I_2=0, I_3=0$, and $I_4=0$ can be decomposed into two branches. Since they are similar, we only discuss one of the cases $I_2=0$, which is equivalent to

$$q(q-pr)(a^2+b^2-c^2)=r(r-pq)(c^2+a^2-b^2).$$

Here we try to find the geometric meaning of this special case. Let R be the radius of circumscribed circle of the triangle ABC. Then

$$\begin{aligned} a^2+b^2-c^2 &= 2ab\cos C = 8R^2\sin A\sin B\cos C \\ c^2+a^2-b^2 &= 2ac\cos B = 8R^2\sin A\sin C\cos B. \end{aligned}$$

We thus have:

$$\frac{r-pq}{q-pr}=\frac{q(a^2+b^2-c^2)}{r(c^2+a^2-b^2)}=\frac{\cos\beta\sin B\cos C}{\cos\gamma\sin C\cos B}=\frac{\tan B\cos\beta}{\tan C\cos\gamma}.$$

Let A_1, B_1, C_1 be the dihedral angles between plane BPA and plane CPA, plane APB and plane CPB, plane APC and plane BPC, respectively. We have,

$$\tan B_1=\frac{\sqrt{2pqr+1-p^2-q^2-r^2}}{\cos\beta-\cos\alpha\cos\gamma},\ \tan C_1=\frac{\sqrt{2pqr+1-p^2-q^2-r^2}}{\cos\gamma-\cos\beta\cos\alpha}.$$

Thus we get,

$$I_2=0 \Longleftrightarrow \frac{\tan B_1}{\tan C_1}=\frac{\tan B\cos\beta}{\tan C\cos\gamma}.$$

Similarly $I_3=0 \Longleftrightarrow \dfrac{\tan C_1}{\tan A_1}=\dfrac{\tan C\cos\gamma}{\tan A\cos\alpha}$ and $I_4=0 \Longleftrightarrow \dfrac{\tan A_1}{\tan B_1}=\dfrac{\tan A\cos\alpha}{\tan B\cos\beta}$.

2.4 Zero Structure for the P3P Equation System

From the previous subsections, we obtain a complete decomposition of $Zero((3)/G)$:

$$Zero((3)/G)=\bigcup_{i=1}^{22} Zero(TS_i/T_iG)$$

where TS_i are polynomial sets in triangular form and T_i are polynomials which could be found in the previous subsections. From this decomposition, we have the following observations.

(1) Since the solutions for each triangular set are well-determined, this decomposition provides a complete set of analytical solutions for the P3P problem.

(2) There is one main component and the rest components correspond to the special cases. The solution of the main component is reduced to solution of one quartic polynomial equation in x^2 and two linear equations, and the special components are reduced to solving quadratic equations (considered as polynomials in x^2 or z^2 if necessary).

(3) From the decomposition, it is easy to see that there are at most four distinct solutions under the reality conditions (2). Notice that this result was proved previously only for the main component.

(4) With the above decomposition, the solving of the P3P problem is reduced to the solving of triangular sets. For detailed study of methods and stability analysis of solving triangular sets, please refer to [18]. Actually, the P3P problem is simpler than the general case. From the decomposition, we may see that we need only to solve linear, quadratic, and quartic equations. Notice that for equations with degree less than or equal to four, we have explicit formulas for their solutions and explicit criteria to decide whether these equations have real or complex solutions[19]. Therefore, for any given values of the six parameters we can use one of the triangular sets to find *all the real solutions* for x, y, z. Since this process does not involve symbolic computation and numerical iteration, it is fast and stable.

3 The Geometric Approach

In this section, we will give some pure geometric criteria for the number of solutions of the P3P problem by conducting a pure geometric analysis.

Let us consider the three conditions $\alpha = \angle APB$, $\beta = \angle APC$, and $\gamma = \angle BPC$ separately. The set of all P satisfying condition $\angle APB = \alpha$ is a toroid S'_{AB}. Similarly, we can define S'_{AC} and S'_{BC}. Because the three toroids are symmetric with the plane ABC, we need only consider what happens on one side of plane ABC. Let S_{AB} denote the half of S'_{AB} which is on one side of plane ABC. We can similarly define S_{AC} and S_{BC}.

We divide the problem into two steps: first, we determine the intersection curve C_A of surfaces S_{AB} and S_{AC}; then, we determine the intersection of C_A with S_{BC}. We have solved the first step completely. For the second step, we have some partial results.

3.1 Determine $\boldsymbol{C}_A = \boldsymbol{S}_{AB} \cap \boldsymbol{S}_{AC}$

First, let us note that under certain conditions (e.g., in Fig.5) C_A may contain a single isolated point A. In this case, we will remove A from C_A and consider C_A as a continuous curve. Let $\widehat{AB}_i$ ($\widehat{AB}_e$) denote the intersection of S_{AB} and plane ABC which is on the same (opposite) side of AB with point C. Since the symmetric axis for toroids S'_{AB} and S'_{AC} meet in point A and point A is also on the toroids, from the shape of the toroid, each branch of C_A must pass through plane ABC. That is, C_A must meet with plane ABC. Curve C_A intersects with plane ABC at, at most, four points: $J = \widehat{AB}_e \cap \widehat{AC}_e$, $H = \widehat{AB}_e \cap \widehat{AC}_i$, $K = \widehat{AB}_i \cap \widehat{AC}_e$, and $I = \widehat{AB}_i \cap \widehat{AC}_i$. From now on, we also use A, B, C to denote the angles of $\angle A, \angle B, \angle C$.

We first give the existence conditions for points J, H, K and I.

• Point J exists if $\beta + \gamma < A$ (Fig.2). In Fig.2, $\angle BJA = \gamma$, $\angle CJA = \beta$, and $\angle BJC = \beta + \gamma$. If $\beta + \gamma$ is large enough, $\widehat{AB}_e$ and $\widehat{AC}_e$ will have no intersection point. If $\beta + \gamma = A$, $\widehat{AB}_e$ is tangent to $\widehat{AC}_e$ at point A. If $\beta + \gamma < A$, the intersection of $\widehat{AB}_e$ and $\widehat{AC}_e$ will exist.

• Point H. There are two cases. If point B is outside of S_{AC}, H_1 exists if $B < \beta$ and $\beta + A < \gamma$. In Fig.3, $\angle BH_1A = \gamma$, $\angle CH_1A = \beta$. In order to ensure the existence of H_1, γ must be greater than $\beta + A$. If $\gamma = \beta + A$, $\widehat{AC}_i$ is tangent to $\widehat{AB}_e$ at point A. Otherwise, point B is inside S_{AC}. We can prove similarly that H_2 exists if $\beta < B$ and $\gamma < \beta + A$.

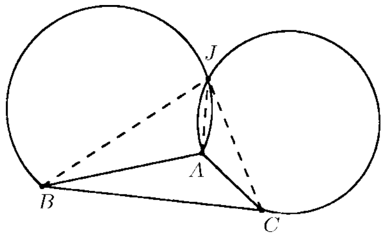

Fig.2. Existence conditions for point J.

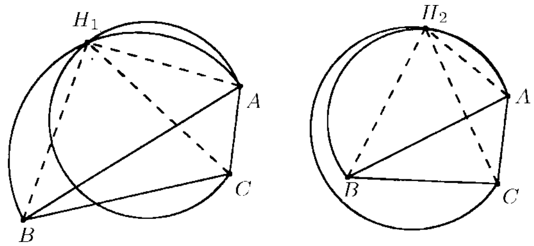

Fig.3. Two cases for point H.

• Point K. There are two cases. If point C is outside of S_{AB}, K_1 exists if $C < \gamma$ and $\gamma + A < \beta$. Otherwise, point C is inside S_{AB}. K_2 exists if $\gamma < C$ and $\beta < \gamma + A$.

• Point I. There are two cases. If point B is outside of S_{AC}, I_1 exists if $B < \beta$, $C < \gamma$, and $\beta + \gamma + A < 2\pi$ (Fig.4). Otherwise, point B is inside S_{AC}. Similar to the first case, I_2 exists if $\beta < B$ and $\gamma < C$ (Fig.4).

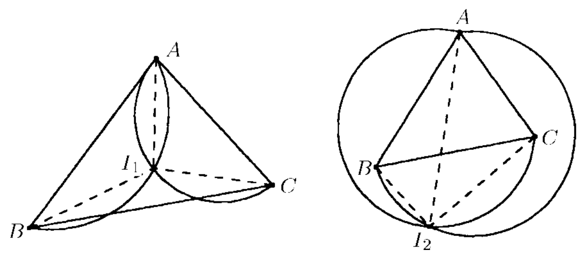

Fig.4. Existence conditions for point I.

We will now give a classification of C_A by counting the intersections of C_A with plane ABC. Suppose that S, U, V are points. We use $E(S)$ ($\overline{E(S)}$) to denote the existence (nonexistence) condition of point S. Notation $S\backslash(U,V)$ means that if S exists then U and V will not exist. Notation $S, U \Rightarrow V$ means that if S and U exist then V exists. From the results in the preceding sections, we have the following results:

$$\begin{cases} J\backslash(K_1,H_1) \\ H_1\backslash(J,K_1,H_2,I_2), H_2\backslash(H_1,I_1) \\ K_1\backslash(J,H_1,K_2,I_2), K_2\backslash(K_1,I_1) \\ I_1\backslash(I_2,K_2,H_2), I_2\backslash(I_1,K_1,H_1) \\ K_2,H_2 \Rightarrow I_2, J, I_2 \Rightarrow H_2, K_2. \end{cases} \tag{R}$$

• C_A intersects plane ABC in four points. Since $H_1\backslash(H_2), K_1\backslash(K_2)$ and $I_1\backslash(I_2)$, point J must exist. From $J\backslash(K_1,H_1)$, K_2 and H_2 must exist. Finally from $K_2, H_2 \Rightarrow I_2$, we get the fourth point I_2. So the four points are J, H_2, K_2, I_2. Then the condition of this case should be $E(J) \cap E(H_2) \cap E(K_2) \cap E(I_2) \cap \overline{E(H_1)} \cap \overline{E(K_1)} \cap \overline{E(I_1)}$, which is equivalent to $E(J) \cap E(I_2)$ by (R). That is,

$$\beta + \gamma < A, \quad \beta < B, \quad \text{and} \quad \gamma < C.$$

In this case, C_A consists of two space curves: one is from point J to I_2 and the other is from H_2 to K_2. Fig.5 shows the case in the plane ABC and a spatial case.

• C_A intersects plane ABC in three points. From $J\backslash(K_1,H_1)$, we know that if J exists, at least either H_2 or K_2 should exist. Actually only one of H_2 and K_2 can exist. Otherwise from $K_2, H_2 \Rightarrow I_2$, we know that there will be four points! Then we know that either H_2 or K_2 exists. From $H_2\backslash(H_1,I_1)$ and $K_2\backslash(K_1,I_1)$, we know that I_2 must exist. Since $J, I_2 \Rightarrow H_2, K_2$, point J must not exist. Since $H_1\backslash(H_2), K_1\backslash(K_2)$ and $I_1\backslash(I_2)$, if we assume that H_1 exists, from $H_1\backslash(J,K_1,H_2,I_2)$ we know that the other two points are K_2 and I_1. This contradicts to $K_2 \backslash (K_1, I_1)$. Thus H_2 must exist. From $H_2 \backslash (H_1, I_1)$ we know the other two points are K_2 and I_2. The condition of this case should be $E(H_2) \cap E(K_2) \cap E(I_2) \cap \overline{E(J)} \cap \overline{E(H_1)} \cap \overline{E(K_1)} \cap \overline{E(I_1)}$. Using (R) we can simplify this condition to $E(H_2) \cap E(K_2) \cap \overline{E(J)}$. That is,

$$|\beta - \gamma| < A < \beta + \gamma, \ B < \beta, \ \text{ and } \ \gamma < C.$$

In this case, C_A consists of two space curves: one is from A to I_2 and the other is from H_2 to K_2. Since the detailed analysis is the same, we will omit them below.

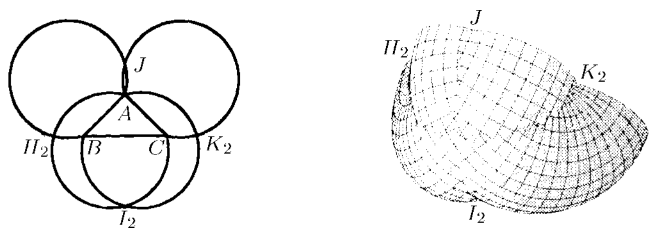

Fig.5. C_A consists of two curves.

• C_A intersects plane ABC in two points. There are five sub-cases.

Case 1. The intersections are J, K_2 (J, H_2) if

$$\begin{cases} \beta + \gamma < A \\ B < \beta \\ \gamma < C \end{cases} \qquad \left(\begin{cases} \beta + \gamma < A \\ \beta < B \\ C < \gamma \end{cases} \right).$$

In this case, C_A consists of one space curve from J to K_2 (J to H_2).

Case 2. The intersections are J, I_1 if

$$\beta + \gamma < A, \quad B < \beta, \ \text{ and } \ C < \gamma.$$

In this case, C_A consists of one space curve from J to I_1.

Case 3. The intersections are H_2, K_1 (H_1, K_2) if

$$\begin{cases} \gamma + A < \beta \\ \beta < B \\ C < \gamma \end{cases} \qquad \left(\begin{cases} \beta + A < \gamma \\ B < \beta \\ \gamma < C \end{cases} \right).$$

In this case, C_A consists of one space curve from H_2 to K_1 (H_1 to K_2).

Case 4. The intersections are K_1, I_1 (H_1, I_1) if

$$\begin{cases} \gamma + A < \beta \\ B < \beta \\ C < \gamma \\ \beta + \gamma + A < 2\pi \end{cases} \qquad \left(\begin{cases} \beta + A < \gamma \\ B < \beta \\ C < \gamma \\ \beta + \gamma + A < 2\pi \end{cases} \right).$$

In this case, C_A consists of one space curve from K_1 to I_1 (H_1 to I_1).

Case 5. The intersections are K_2, I_2 (H_2, I_2) if

$$\begin{cases} \beta + A < \gamma \\ \beta < B \\ \gamma < C \end{cases} \qquad \left(\begin{cases} \gamma + A < \beta \\ \beta < B \\ \gamma < C \end{cases} \right).$$

In this case, C_A consists of one space curve from K_2 to I_2 (H_2 to I_2).

• C_A intersects plane ABC in one point. We need to consider two sub-cases.

Case 1. The intersection is H_2 (K_2) if

$$\begin{cases} |\beta-\gamma| < A < \beta+\gamma \\ \beta < B \\ C < \gamma \end{cases} \qquad \left(\begin{cases} |\beta-\gamma| < A < \beta+\gamma \\ B < \beta \\ \gamma < C \end{cases} \right).$$

In this case, C_A consists of one space curve from A to H_2 (A to K_2).

Case 2. The intersection is I_1 if

$$|\beta-\gamma| < A < \beta+\gamma, B < \beta, C < \gamma, \text{ and } \beta+\gamma+A < 2\pi.$$

In this case, C_A consists of one space curve from A to I_1.

3.2 Determine $\boldsymbol{C}_A \cap \boldsymbol{S}_{BC}$

We should comment that the seemingly tedious analysis in the preceding section is actually based on strong geometric intuition coming from a dynamic geometry software: *Geometry Expert*[20]. Using *Geometry Expert*, we can see clearly how $\widehat{AB_i}$ and $\widehat{AB_e}$ change when changing the six free parameters continuously. But for the 3D case, there is still no adequate software for us to get an intuitive idea of how C_A looks like. Here are some partial results we got.

Lemma 2. *The P3P problem has one or three solutions if C_A consists of one space curve and the two intersection points of plane ABC and C_A are not on the same side of S_{BC}.*

Proof. Since C_A is a continuous space curve and the two intersection points of plane ABC and C_A are not in the same side of S_{BC}, C_A must intersect S_{BC} for odd times. In addition, the maximal number of solution is four, hence the problem has a unique solution or three solutions. □.

Lemma 3. *If β, γ $(\alpha,\beta;\gamma,\alpha)$ are obtuse angles and $\alpha > A$ $(\beta > B;\ \gamma > C)$, then the P3P problem can only have one or three solutions.*

Proof. See Fig.6. We have $\angle BI_1A = \gamma > \frac{\pi}{2}$, $\angle CI_1A = \beta > \frac{\pi}{2}$. Point I_1 is on the same side of BC with point A. According to the "reality conditions", we know that $\alpha+\beta+\gamma < 2\pi$, which implies that point I_1 is inside S_{BC}. Condition $\alpha > A$ means that point A is in the outside of S_{BC}. Thus the result follows according to above lemma. □

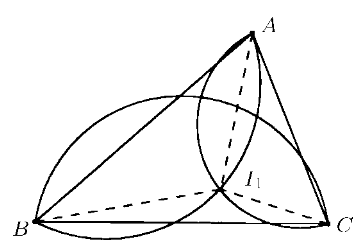

Fig.6. A unique solution exists.

Theorem 4. *Under the reality conditions* (2), (1) *if β, α, and γ are obtuse, then the P3P problem can only have one solution;* (2) *furthermore if $A < \alpha, B < \beta, C < \gamma$, then the P3P problem has a unique solution.*

Proof. From Lemma 3, we know that the problem will have one or three solutions since β, α, and γ are obtuse and at least one of A, B, C is acute. Since the three angles are all obtuse, the three toroids and their intersection curves are concave. This implies that they can only have one intersection point. If $A < \alpha, B < \beta, C < \gamma$, from Subsection 3.1, point I_1 must exist. Similar to Lemma 3, points A and I_1 must be on different sides of S_{BC}. Similar to the proof of Lemma 2, a solution must exist. □

4 Complete Solution of Two Special Cases

4.1 Case of $\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{c}$ and $\boldsymbol{r}=\boldsymbol{q}$

Without loss of generality, let $a=b=c=1$. The P3P equation system becomes

$$\begin{cases} y^2+z^2-2yzp-1=0 \\ z^2+x^2-2zxq-1=0 \\ x^2+y^2-2xyq-1=0. \end{cases} \qquad (PS_1)$$

Using Wu-Ritt's method, this equation system has the following two components:

$$\begin{cases} f_1 = 2(p-1)x^2+4q(1-p)zx+1-2p \\ f_2 = y-z \\ f_3 = 2(p-1)z^2+1 \end{cases} \qquad (ES_1)$$

$$\begin{cases} g_1 = 2qC_1x+z(2z^2C_2-1-8pq^2) \\ g_2 = C_1y+2z(z^2C_2-C_3) \\ g_3 = 4C_2^2z^4-4C_3C_2z^2+C_1^2. \end{cases} \qquad (ES_2)$$

Here $C_1=4q^2-1$, $C_2=1-2q^2+p$, $C_3=pC_1+C_2=4pq^2+1-2q^2$.

Let points I,J,H,K be defined as in Section 3. If (ES_1) has positive solutions, these solutions must be the intersection points of curve AI or JI and S_{AB}. Since $p<1$, $1+(2p-2)z^2$ always has one positive solution $z=\sqrt{\frac{1}{2-2p}}$, so does y. The number of positive solutions of (ES_1) is determined by $f_1(x)=0$. Notice that $p<1$ and $f_1(x)$ is a quadratic equation in x, we have the following results.

- Equation system (ES_1) has one positive solution if and only if

$$\begin{cases} q>0 \\ p=\dfrac{1+q^2}{2} \text{ or } \dfrac{1}{2} \end{cases} \quad \text{or} \quad p<\frac{1}{2}.$$

- Equation system (ES_1) has two positive solutions if and only if

$$q>0 \quad \text{and} \quad \frac{1+q^2}{2}>p>\frac{1}{2}.$$

Now we discuss (ES_2). If (ES_2) has positive solutions, these solutions must be the intersection points of curve HK and S_{AB}. From the "reality conditions" (2) we know that the coefficient of z^4 in g_3, $C_2=1-2q^2+p$, won't vanish. In addition, a necessary condition for H and K to exist is $\beta<\frac{\pi}{3}$, so $q>\frac{1}{2}$. Thus, both C_1 and C_2 are not zero.

We can prove that if (x,y,z) is one positive solution of (ES_2), (x,z,y) will be another positive solution of (ES_2). We also proved that if z is positive, then g_1 always has a positive solution for x[14]. So the number of positive solutions of (ES_2) is equal to the number of positive solutions of g_3. Let $g(w)$ be obtained by replacing z^2 in g_3 by w. Denote the discriminant of $g(w)$ by Δ_2. We have

$$\Delta_2 = 64q^2(1+2p)(2pq^2+1-3q^2)(1-2q^2+p)^2.$$

From $\beta<\frac{\pi}{3}$ and $\alpha<2\beta$, we know $q>\frac{1}{2}$, $p>-\frac{1}{2}$ and $1-2\,q^2+p\neq 0$. Thus the sign of Δ_2 is determined by $2\,pq^2+1-3\,q^2$. Using Wu-Ritt's method again, we can prove that if $\Delta_2=0$, equation system (PS_1) has no solution. So $g(w)$ has two real solutions if and only if $2pq^2+1-3\,q^2>0$, and the two real solutions are all positive[14]. Thus (ES_2) has two positive solutions if and only if

$$q>\frac{1}{2} \text{ and } p>\frac{3}{2}-\frac{1}{2q^2}.$$

Otherwise (ES_2) has no positive solution.

We still need to consider the reality conditions (2): $0 < \alpha, \beta < \pi$, $0 < \alpha + 2\beta < 2\pi$, and $2\beta > \alpha$, which can be reduced to

$$-1 < p < 1, -1 < q < 1, 2q^2 - 1 < p.$$

Combining the above conditions, we have the following classification for the P3P problem.

1) Point P has four solutions, if and only if

$$1 > q > \frac{1}{2}, \quad \frac{1+q^2}{2} > p > \frac{1}{2}, \quad \text{and} \;\; p > \frac{3}{2} - \frac{1}{2q^2}.$$

2) Point P has three solutions, if and only if

$$\begin{cases} \frac{1}{2} < q < \frac{\sqrt{2}}{2} \\ \frac{3}{2} - \frac{1}{2q^2} < p \le \frac{1}{2} \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{2} < q < 1 \\ p = \frac{1+q^2}{2}. \end{cases}$$

3) Point P has two solutions, if and only if

$$\begin{cases} 0 < q \le \frac{1}{2} \\ \frac{1}{2} < p < \frac{1+q^2}{2} \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{2} < q < 1 \\ \frac{1+q^2}{2} < p < 1 \end{cases} \quad \text{or} \quad \begin{cases} \frac{\sqrt{2}}{2} < q < 1 \\ \frac{1}{2} < p \le \frac{3}{2} - \frac{1}{2q^2} \;\; \text{and} \;\; p > 2q^2 - 1. \end{cases}$$

4) Point P has one solution, if and only if

$$\begin{cases} -\frac{\sqrt{3}}{2} < q < \frac{1}{2} \;\text{or}\; \frac{\sqrt{2}}{2} \le q < \frac{\sqrt{3}}{2} \\ 2q^2 - 1 < p < \frac{1}{2} \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{2} < q < \frac{\sqrt{2}}{2} \\ 2q^2 - 1 < p \le \frac{3}{2} - \frac{1}{2q^2} \end{cases}$$

$$\text{or} \quad \begin{cases} 0 < q \le \frac{1}{2} \;\text{or}\; \frac{\sqrt{2}}{2} \le q < \frac{\sqrt{3}}{2} \\ p = \frac{1}{2} \end{cases} \quad \text{or} \quad \begin{cases} 0 < q \le \frac{1}{2} \\ p = \frac{1+q^2}{2}. \end{cases}$$

Fig.7 is the solution distribution diagram for this special case. L_1 is $p = \frac{1+q^2}{2}$, L_2 is $p = 2q^2 - 1$, and L_3 is $p = \frac{3}{2} - \frac{1}{2q^2}$. Table 1 shows where the solutions come from for each region.

Here are some general observations from Fig.7. The P3P problem most probably will have one solution. The probabilities of hoving two, three and four solutions decrease in order. The P3P problem tends to have more number of solutions when the three perspective angles are small. The most complicated cases occur when the three perspective angles are almost equal to the correspondent inner angles of triangle ABC.

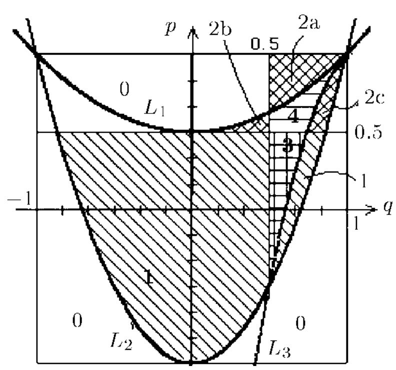

Fig.7. Solution distribution for case $a = b = c = 1, q = r$.

Table 1

Solution number	1	2a	2b	2c	3	4
Equation systems	ES_1	ES_2	ES_1	ES_1	ES_1 and ES_2	ES_1 and ES_2

4.2 Case of $b = c$ and $r = q$

Without loss of generality, we may assume that $b = c = 1$. In this case, there are three parameters p, q and a. From the cosine theorem we know that $a = \sqrt{2 - 2\cos A}$. We will use $\cos A$ as the third parameter. The equation system becomes

$$\begin{cases} y^2 + z^2 - 2yzp - 2 + 2\cos A = 0 \\ z^2 + x^2 - 2zxq - 1 = 0 \\ x^2 + y^2 - 2xyq - 1 = 0. \end{cases} \tag{PS_2}$$

Similar to the first example, we get the following two components:

$$\begin{cases} (p-1)x^2 + 2q(1-p)zx - p + \cos A \\ y - z \\ 2(p-1)z^2 + 1 - \cos A. \end{cases} \tag{ES_3}$$

$$\begin{cases} 2qC_1x - z(4q^2(\cos A + p) - z^2C_2 - C_1) \\ C_1y - z(4q^2(\cos A + p) - z^2C_2 - 2C_1) \\ C_2^2z^4 - 2C_2C_3z^2 + C_1^2. \end{cases} \tag{ES_4}$$

Here $C_1 = 2q^2 - 1 + \cos A$, $C_2 = 1 - 2q^2 + p$, $C_3 = 2\,pq^2 + (1 - \cos A)\,(1 - 2q^2)$. The structure of ($ES_3$) and ($ES_4$) are similar to that of (ES_1) and (ES_2). Hence, we have

• Equation system (ES_3) has one positive solution if and only if

$$\begin{cases} q > 0 \\ p = \cos A + \dfrac{q^2}{2} \text{ or } \cos A \end{cases} \quad \text{or} \quad p < \cos A.$$

• Equation system (ES_3) has two positive solutions if and only if

$$q > 0 \quad \text{and} \quad \cos A < p < \cos A + \frac{q^2}{2}.$$

• Equation system (ES_4) may not have one positive solution, and it has two positive solutions if and only if

$$q > \sqrt{\frac{1 - \cos A}{2}} \quad \text{and} \quad p > 2 - \cos A + \frac{\cos A - 1}{2q^2}.$$

Now, the classification of the solutions can be obtained easily.

5 Conclusion

By decomposing the equation system for the P3P problem into triangular sets with WU-Ritt's zero decomposition method, we gave fast, stable, and complete methods for solving the P3P problem. Although we gave some pure geometric criteria for the solution classification of the P3P problem, this problem still resists a complete solution and should be further studied. In the algebraic computation approach, the main problem is to find a set of simple criteria. In the geometric approach started in this paper, we need to find criteria of larger scope.

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References

[1] Abidi M A, Chandra T. A new efficient and direct solution for pose estimation using quadrangular targets: Algorithm and evaluation. *IEEE Transaction on Pattern Analysis and Machine Intelligence,* May 1995, 17(5): 534–538.

[2] Fischler M A, Bolles R C. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartomated cartography. *Communications of the ACM,* June 1981, 24(6): 381–395.

[3] Horaud R, Conio B, Leboulleux O. An analytic solution for the perspective 4-point problem. *Computer Vision, Graphics, and Image Processing,* 1989, 47: 33–44.

[4] Su C, Xu Y, Li H, Liu S. Wu's method's application in computer animation. In *The Fifth Int. Conf. CAD/CG,* China, Vol.1, 1997, pp.211–215.

[5] Yuan J S C. A general photogrammetric method for determining object position and orientation. *IEEE Transactions on Robotics and Automation,* April 1989, 5(2): 129–142.

[6] Wolfe W J, Mathis D, Weber C, Magee M. The perspective view of three points. *IEEE Transactions on Pattern Analysis and Machine Intelligence,* Jan. 1991, 13(1): 66–73.

[7] Hung Y, Yeh P, Harwood D. Passive ranging to known planar points sets. In *Proc. IEEE Int. Conf. Rob. and Auto.*, St. Louis, Vol.1, 1985, pp.80–85.

[8] Haralick R M, Lee C, Ottenberg K, Nolle M. Analysis and solutions of the three-point perspective pose estimation problem. In *Proc. of the Int. Conf. Computer Vision and Pattern Recognition,* 1991.

[9] DeMenthon D, Davis L S. Exact and approximate solutions of the perspective-three-point. *IEEE Transaction on Pattern Analysis and Machine Intelligence,* Nov. 1992, 14(11): 1100–1105.

[10] Wolfe W J, Jones K. Camera calibration using the perspective view of a triangle. In *Proc. SPIE Conf. Auto. Inspection Measurement,* Vol.730, Cambridge, 1986.

[11] Su C, Xu C Y, Li H, Liu S. Necessary and sufficient condition of positive root number of P3P problem. *Chinese Journal of Computers,* Dec. 1998, 21(12): 1084–1095. (in Chinese)

[12] Yang L. A simplified algorithm for solution classification of the perspective-three-point problem. *MM-Preprints,* 1998, 17: 135–145.

[13] Wu W T. Basic Principles of Mechanical Theorem Proving in Geometries, Volume I: Part of Elementary Geometries. Beijing: Science Press, 1984 (in Chinese), English Version, Springer-Verlag, 1995.

[14] Gao X S, Cheng H F. On the solution classification of the "P3P" problem. In *Proceedings of the Third Asian Symposium on Computer Mathematics,* Li Z B (ed.), LanZhou University Press, 1998, pp.185–200.

[15] Collins G E. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *LNCS,* Vol. 33, Springer-Verlag, 1975, pp.134–183.

[16] Mishra B. Algorithmic Algebra. New York: Springer-Verlag, 1993.

[17] Buchberger B, Collins G E, Loos R. Computer Algebra Symbolic and Algebraic Computation. Berlin: Springer-Verlag, 1988.

[18] Gao X S, Chou S C. On the theory of resolvents and its applications. *Sys. Sci. and Math. Sci.,* 1999, 12(Suppl.): 17–30.

[19] Arnon D S. Geometric reasoning with logic and algebra. *Artificial Intelligence,* 1988, 37(1-3): 37–60.

[20] Gao X S, Zhang J Z, Chou S C. Geometry Expert. Taipai: Nine Chapter Pub., Taiwan, 1998. (in Chinese)

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