Belief Revision by Sets of Sentences

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Abstract

The aim of this paper is to extend the system of belief revision developed by Alchourrón, Gärdenfors and Makinson (AGM) to a more general framework. This extension enables a treatment of revision not only by single sentences but also by any sets of sentences, especially by infinite sets. The extended revision and contraction operators will be called general ones, respectively. A group of postulates for each operator is provided in such a way that it coincides with AGM's in the limit case. A notion of the nice-ordering partition is introduced to characterize the general contraction operation. A computation-oriented approach is provided for belief revision operations.

Keywords: Belief revision, the logic of theory change, epistemic entrenchment, default logic.

1 Introduction

In recent years, there has been much concern in artificial intelligence, database theory and philosophical logic about the problem of *Belief Revision* or *Theory Change*. Roughly, Belief Revision is processes or computer programs of incorporating new information into a K(nowledge) B(ase) while preserving consistency. The most well-known system of this sort is developed by Alchourrón, Gärdenfors and Makinson(1985, 1988), which we refer to the *AGM theory* (see [1,10,11]). Although studies along the AGM approach have reached a new degree of sophistication through a number of different proposals put forth by several researchers in later years, there are still a few problems in belief revision remained to investigate further.

1. **Generalization:** The original intention of belief revision is a treatment of revising a belief set (or a knowledge base) by a set of sentences (new information). But AGM's work is mostly limited to the case where belief sets are revised by single sentences. There have already been several investigations into revision by sets of sentences (see [8,22,13,14,9]). However, to my knowledge, it seems that a general framework for the belief revision by sets of sentences, especially by infinite sets, has not yet been formed. In this paper, we will try to explore a few ways to deal with the problem.

   Formally, let K be a belief set, A an arbitrary sentence and F an arbitrary set of sentences. In the AGM framework, there are three basic operations for belief changes:

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expansion, contraction and revision, denoted by $K+A$, $K-A$ and $K*A$, respectively. These operations can only deal with the changes of belief set K by a single sentence A. We should extend them to cope with the case that a belief set is changed by any set of sentences. For the expansion, we can easily define a *general expansion* operator as $K+F=Cn(K\cup F)$ just as AGM did for single sentence. For the revision, according to AGM's context, revising a belief set K by a set of new information F should mean the process of deleting some of the sentences from the belief set K and adding F to it in such a way that the result is a consistent belief set. In order to formalize the idea, we will introduce a set of postulates to specify the general revision operation.

The generalization of contraction seems to be more difficult if we want to follow the AGM approach strictly. In AGM's system, contracting K with sentence A means removing A and some sentences from K so that the resulting belief set cannot imply A. This idea is not easily extended to the general case. For example, let $F=\{A,B\}$ and denote the contraction of K with F by $K\ominus F$. If we still identify F with the conjunction of all sentences of F, we should have $K\ominus F=K-(A\wedge B)$. But $K-(A\wedge B)$ does not mean retracting both of A and B from K. On the other hand, the connections between revision and contraction which AGM have established by employing *Harper* and *Levi identity* lost in the infinite case. If we interpret contracting K with F as removing some sentences of K so that the remained set is closed and consistent with F, then we will have most properties of contraction which we expect. In Section 3 we will give some postulates to characterize the general contraction operation and establish connections between the general revision and contraction operation.

2. **Fundamentality:** AGM (1988) viewed the behaviors of belief revision as arising from a more fundamental conception, that of *Epistemic entrenchment.* Fortunately, this conception has been accepted by more and more researchers and has received much concern in AI, which led to a number of different proposals for different applications (see [7,16,6,12]). However, there seem to be some drawbacks with the epistemic entrenchment.

   i) It is unclear that why or when a sentence A has a higher degree of epistemic entrenchment than a sentence B. It seems that there is a more basic concept than epistemic entrenchment, that of *degree of belief.* It is self-evident that even if all sentences in a belief set are accepted by an agent or a computer system, different sentences may be believed with different degrees of belief. Furthermore, the degree of belief of a sentence should be a pure subjective concept, which depend only upon how strong the agent believes it. In Section 5 we try to introduce a structure, called *total-ordering partition* characterizing this idea. Intuitively, we can divide our believes into finite or infinitely many groups according to the degrees in which we believe them, and then arrange these groups into a total-ordering. This kind of arrangement (or partition) is logically independent. Another point is that a reasonable agent will not satisfy a first-sight intuition, or at any rate he will place logically equivalent sentences into the same group after he concluded the equivalence of them. So, at the next step, we will rearrange the partition above so that it satisfies some logical constraints. The following restriction seems to be rational:

   If $A_1,\cdots,A_n\vdash A$, the degree of belief of A is not less than the minimum of degree of belief of $A_1,\cdots,A_n$

   That is to say, if we have believed all of the premises of an inference, we should immediately believe its consequence. A total-ordering partition by rearranging with logical constraint will be called a *nice-ordering partition.* The process of rearrangements can be viewed as the systematization of the agent's original belief by using the logical tool.

Although in Section 6 we will show that a nice-ordering partition is equivalent to an order of epistemic entrenchment in some sense, the notion of nice-ordering partition seems to have a better intuitive sense than that of epistemic entrenchment. In Section 7, we provide a concrete contraction operator, written $\dot{\ominus}$, generated by a nice-ordering partition of a belief set which satisfies all postulates for the general contraction.

ii) There is a computational drawback with epistemic entrenchment. It is a question how we can obtain an order of epistemic entrenchment for any given belief set. In Section 5 we will try to give a constructive approach in which we can arrive at a nice-ordering partition of a belief set from a well-ordering partition of its base. The resulting partition is also well-ordered. We will call it *a nice-well-ordering partition.*

3. **Computationality:** Although AGM provided three methods of specifying revision and contraction operations, none of them are 'computation-friendly' or constructive. In Section 8, we try to explore a way in which a constructive contraction operator[1], written as $\hat{\ominus}$, can be generated by a nice-well-ordering partition, which will be useful even in the case that K and F are finite. This constructive approach can also give us a better understanding of belief revision: *belief revision* is a choice that a cautious man make in order to preserve consistency and retain as many information which he believes most as possible.

Finally, as an application of our approach, we provide a solution to a problem left open by Nebel(1992) in [21], where Nebel showed that there exists a tight connection between belief revision and the so-called *ranked default theories* in the finite case. We will show that it is also true in the infinite case.

2 Preliminaries

Throughout this paper, we consider first-order language L with the standard logical connectives $\neg, \vee, \wedge, \rightarrow$ and $\leftrightarrow$, denote individual sentences by A, B, or C, and denote sets of sentences by K, F, Γ, Δ ect. If $F = \{A_1, \cdots, A_n\}$ is a set of sentences, $\wedge F$ means $A_1 \wedge \cdots \wedge A_n$. We shall assume that the underlying logic includes classical first-order logic and that it is compact. The notation $\vdash$ means classical first-order derivability and Cn the corresponding closure operator, i.e.

$$Cn(K) = \{A \in L | K \vdash A\}$$

We call a set K of sentences as a *belief set*, which means that $K = Cn(K)$, usually denoted by K or K'. The notation $K_\perp$ denotes the belief set which is inconsistent. A set Γ of sentences is closed also means $\Gamma = Cn(\Gamma)$. As usual $K * A$ denotes the revision of the belief set K by a sentence A, and $K - A$ the contraction of K with sentence A in the AGM's framework. The notation $K + F$ denotes $Cn(K \cup F)$.

3 Postulates for General Revision and Contraction

In order to formalize revision and contraction operations, AGM introduced sets of rationality postulates for these operations. We repeat them here for the sake of comparing our postulates with AGM's.

[1] Here "constructive" just means "theoretical constructive".

The postulates for revision operator '$*$':

(*1) $K * A = Cn(K * A)$.
(*2) $A \in K * A$.
(*3) $K * A \subseteq K + A$.
(*4) If $\neg A \notin K$, then $K + A \subseteq K * A$.
(*5) $K * A = K_\perp$ if and only if $\vdash \neg A$.
(*6) If $\vdash A \leftrightarrow B$, then $K * A = K * B$.
(*7) $K * (A \wedge B) \subseteq (K * A) + B$.
(*8) If $\neg B \notin K * A$, then $(K * A) + B \subseteq K * (A \wedge B)$.

The postulates for contraction operator '$-$':

($-$1) $K - A = Cn(K - A)$.
($-$2) $K - A \subseteq K$.
($-$3) If $A \notin K$, then $K - A = K$.
($-$4) If $\nvdash A$, then $A \notin K - A$.
($-$5) $K \subseteq (K - A) + A$
($-$6) If $\vdash A \leftrightarrow B$, then $K - A = K - B$.
($-$7) $(K - A) \cap (K - B) \subseteq K - (A \wedge B)$.
($-$8) If $A \notin K - (A \wedge B)$, then $K - (A \wedge B) \subseteq K - A$.

The intuition underlying their postulates is that 'changes should be minimal, so that when changing beliefs in response to new evidence, one should continue to believe as many of the old beliefs as possible'. With the same motivation, we will first introduce a set of postulates for the general revision operation.

Let K be a belief set and F be an arbitrary set of sentences in L. Revising K with F, denoted by $K \oplus F$, means deleting some sentences from K in order that the resulting belief set is consistent with F, and then adding F to it.

More formally, we call $\oplus$ a general revision function if it satisfies the following eight postulates:

($\oplus$1) $K \oplus F = Cn(K \oplus F)$.
($\oplus$2) $F \subseteq K \oplus F$.
($\oplus$3) $K \oplus F \subseteq K + F$.
($\oplus$4) If $K \cup F$ is consistent, then $K + F \subseteq K \oplus F$.
($\oplus$5) $K \oplus F = K_\perp$ iff F is inconsistent.
($\oplus$6) If $Cn(F_1) = Cn(F_2)$, then $K \oplus F_1 = K \oplus F_2$.
($\oplus$7) $K \oplus (F_1 \cup F_2) \subseteq K \oplus F_1 + F_2$.
($\oplus$8) If $F_2 \cup (K \oplus F_1)$ is consistent, then $(K \oplus F_1) + F_2 \subseteq K \oplus (F_1 \cup F_2)$.

It is not difficult to see that the postulates above are direct generalizations of those of AGM's. In fact, if let

$$K * A \overset{Def}{=} K \oplus \{A\}$$

and identify $\{A, B\}$ with $A \wedge B$, then two sets of postulates are identical. Thus AGM's revision can be conceived as a special case of the general revision. To mark off them, we will call the former as *single revision*.

By comparing our postulates with AGM's, we will find that postulates for the general revision operation is no less nature than those for single revision, especially for ($\oplus$7) and ($\oplus$8).

As was mentioned in Section 1, the generalization of contraction operations is more difficult than revision. If we followed AGM's idea, that is, viewing the contraction of K with F as the result of removing F and some sentences from K, we would lost the connection between revision and contraction in the case that F is a infinite set. In fact, a more general meaning of contraction operations is: removing some sentences of K in such a way that the resulting belief set is consistent with F.

More precisely, we call $\ominus$ a general contraction function if it satisfies the following eight postulates:

($\ominus$1) $K \ominus F = Cn(K \ominus F)$.
($\ominus$2) $K \ominus F \subseteq K$.
($\ominus$3) If $F \cup K$ is consistent, then $K \ominus F = K$.
($\ominus$4) If F is consistent, then $F \cup K \ominus F$ is consistent; otherwise $K \ominus F = K$.
($\ominus$5) $\forall A \in K(F \vdash \neg A \rightarrow K \subseteq K \ominus F + A)$.
($\ominus$6) If $Cn(F_1) = Cn(F_2)$, then $K \ominus F_1 = K \ominus F_2$.
($\ominus$7) $K \ominus F_1 \subseteq K \ominus (F_1 \vee F_2) + F_1$.
($\ominus$8) If $F_1 \cup (K \ominus (F_1 \vee F_2))$ is consistent, then $K \ominus (F_1 \vee F_2) \subseteq K \ominus F_1$.
Here $F_1 \vee F_2 = \{A \vee B : A \in F_1 \wedge B \in F_2\}$[2].

The intuition behind the postulates ($\ominus$1)–($\ominus$4) and ($\ominus$6) are obvious. ($\ominus$5) is the generalization of Gärdenfors' *Principle of Recovery*. The final two postulates are somewhat counterintuitive. In fact, We use contraction just as an auxiliary tool to construct revision.

Observation 3.1. *Let*

$$K - A \stackrel{Def}{=} K \ominus \{\neg A\} \tag{1}$$

If function '$\ominus$' satisfies ($\ominus$1) *to* ($\ominus$8), *'$-$' will satisfy* (-1) *to* (-8).

Proof. (-1) to (-6) follow ($\ominus$1) to ($\ominus$6) directly. (-8) follows Eq.(1) and ($\ominus$8). To show (-7), let $C \in K - A \cap K - B$, that is, $C \in K \ominus \{\neg A\} \cap K \ominus \{\neg B\}$, or $C \in K \ominus \{\neg A\}$ and $C \in K \ominus \{\neg B\}$. By ($\ominus$2) $C \in K$, it follows from ($\ominus$7) that $C \in K \ominus (\{\neg A\} \vee \{\neg B\}) + \{\neg A\}$ and $C \in K \ominus (\{\neg A\} \vee \{\neg B\}) + \{\neg B\}$. Hence $(A \wedge B) \vee C \in K \ominus (\{\neg A\} \vee \{\neg B\})$, or $(A \wedge B) \vee C \in K \ominus (\{\neg A \vee \neg B\})$, i.e., $(A \wedge B) \vee C \in K - (A \wedge B)$. On the other hand, ($-5$) implies $\neg(A \wedge B) \vee C \in K - (A \wedge B)$. Thus $C \in K - (A \wedge B)$, as desired. □

Observation 3.2. i) $K \oplus F = K \oplus Cn(F)$. ii) $K \ominus F = K \ominus Cn(F)$.

Proof. Straightforward from ($\oplus$6) and ($\ominus$6). □

4 From Contraction to Revision and Vice Versa

A nature question now is whether either general contraction or revision can be defined in terms of the other. Before answer the question we present a lemma.

Lemma 4.1.
i) *If function* $\oplus$ *satisfies* ($\oplus$1)–($\oplus$4), *then* $K \oplus F = ((K \oplus F) \cap K) + F$.
ii) *If function* $\ominus$ *satisfies* ($\ominus$1)-($\ominus$5), *then* $K \ominus F = (K \ominus F + F) \cap K$.

[2] It is not difficult to show that $F_1 \vee F_2 \dashv\vdash Cn(F_1) \cap Cn(F_2)$

Proof. i) Since $(K\oplus F)\cap K \subseteq K\oplus F$ and $F \subseteq K\oplus F$, by $(\oplus 1)$, $((K\oplus F)\cap K)+F \subseteq K \oplus F$. If $K \cup F$ is consistent, we have

$$K \oplus F = K + F \subseteq (K+F)\cap K + F = (K\oplus F)\cap K + F$$

If not so, there exists $A \in K$ such that $F \vdash \neg A$, and therefore there is a finite subset $\bar{F}$ of F such that $\bar{F} \vdash \neg A$, so $A \vdash \neg(\wedge \bar{F})$. For any $B \in K \oplus F$, by $(\oplus 1)$, $\neg(\wedge\bar{F})\vee B \in K\oplus F$. We also have $\neg(\wedge\bar{F})\vee B \in K$. Then $\neg(\wedge\bar{F})\vee B \in (K\oplus F)\cap K$, so $B \in ((K\oplus F)\cap K)+F$. Thus $K\oplus F \subseteq ((K\oplus F)\cap K)+F$.

ii) It is obvious that $K\ominus F \subseteq ((K\ominus F)+F)\cap K$. For any $B \in ((K\ominus F)+F)\cap K$, $B \in K$ and $B \in K\ominus F+F$. So there is a finite subset $\bar{F}$ of F such that $\neg(\wedge\bar{F})\vee B \in K\ominus F$. If $F\cup K$ is consistent, then we have $B \in K\ominus F$ obviously. If not, there is $A \in K$ such that $F \vdash \neg A$. Suppose that $\bar{F}' \subseteq F$ such that $\bar{F} \subseteq \bar{F}'$ and $\bar{F}' \vdash \neg A$. Then $\neg(\wedge\bar{F})\vee B \in K\ominus F$ implies that $\neg(\wedge\bar{F}')\vee B \in K\ominus F$, and $A \in K$ will imply that $\neg(\wedge\bar{F}') \in K$. So by $(\ominus 5)$, $K = K\ominus F + \neg(\wedge\bar{F}')$. It follows from $B \in K$ that $(\wedge\bar{F}')\vee B \in K\ominus F$. Hence $B \in K\ominus F$, i.e., $(K\ominus F+F)\cap K \subseteq K\ominus F$. □

Similar to *Levi identity* in AGM's framework, we define the revision function by the contraction function as follows:

(Def $\oplus$) $$K\oplus F = (K\ominus F)+F$$

Theorem 4.1. *If the contraction function $\ominus$ satisfies $(\ominus 1)$ - $(\ominus 8)$, then the revision function $\oplus$ obtained from (Def$\oplus$) satisfies $(\oplus 1)$–$(\oplus 8)$ as well as $K\ominus F = (K\oplus F)\cap K$.*

Proof. Postulates $(\oplus 1)$–$(\oplus 2)$ are trivial. $(\oplus 3)$ follows from $(\ominus 2)$ and $(\oplus 4)$ follows from $(\ominus 3)$. For $(\oplus 5)$, assume F is inconsistent. Then $K\ominus F = K$ from $(\ominus 4)$, so $K\oplus F = K\ominus F+F = K+F = K_\perp$ by inconsistency of F. For the converse, assume $K\oplus F = K_\perp$, i.e. $K\ominus F+F$ is inconsistent, so is $F\cup K\ominus F$. By $(\ominus 4)$, F is inconsistent. Postulate $(\oplus 6)$ follows directly from $(\ominus 6)$.

To show $(\oplus 7)$, let $B \in K\oplus(F_1\cup F_2)$, i.e., $B \in K\ominus(F_1\cup F_2)+(F_1\cup F_2)$, then there exists finite subsets $\bar{F}_1, \bar{F}_2$ of F_1 and F_2, respectively, such that $\neg(\wedge\bar{F}_1)\vee\neg(\wedge\bar{F}_2)\vee B \in K\ominus(F_1\cup F_2)$. It is not difficult to show that $Cn(F_1) = Cn((F_1\cup F_2)\vee F_1)$, so, by postulate $(\ominus 6)$ and $(\ominus 7)$, we have

$$\begin{aligned} K\ominus(F_1\cup F_2) &\subseteq K\ominus((F_1\cup F_2)\vee F_1)+(F_1\cup F_2) \\ &= K\ominus F_1+(F_1\cup F_2) \end{aligned}$$

Hence $\neg(\wedge\bar{F}_1)\vee\neg(\wedge\bar{F}_2)\vee B \in K\ominus F_1+(F_1\cup F_2)$, therefore $B \in K\ominus F_1+(F_1\cup F_2)$, i.e., $B \in K\oplus F_1+F_2$.

In order to show $(\oplus 8)$, suppose that $F_2\cup K\oplus F_1$ is consistent, or $F_1\cup F_2\cup K\ominus F_1$ is consistent. Note that $Cn(F_1) = Cn((F_1\cup F_2)\vee F_1)$. So $(F_1\cup F_2)\cup K\ominus((F_1\cup F_2)\vee F_1)$ is consistent, hence, by $(\ominus 6)$and $(\ominus 8)$, $K\ominus F_1 = K\ominus((F_1\cup F_2)\vee F_1) \subseteq K\ominus(F_1\cup F_2)$. We conclude that $K\oplus F_1+F_2 \subseteq K\oplus(F_1\cup F_2)$.

$K\ominus F = (K\oplus F)\cap K$ follows from Lemma 4.1 directly. □

Similar to *Harper identity* , we can define contraction function by revision function as follows:

(Def $\ominus$) $$K\ominus F = (K\oplus F)\cap K$$

Theorem 4.2. *If the revision function $\oplus$ satisfies $(\oplus 1)$–$(\oplus 8)$, then the contraction function $\ominus$ obtained from $(Def\ominus)$ satisfies $(\ominus 1)$–$(\ominus 8)$ as well as $K \oplus F = (K \ominus F) + F$.*

Proof. Postulates $(\ominus 1)$–$(\ominus 4)$ follow from $(\oplus 1)$–$(\oplus 5)$, and $(\ominus 6)$ follows directly from $(\oplus 6)$ by means of (Def $\ominus$).

To show $(\ominus 5)$, assume that $A \in K$. If $F \vdash \neg A$, then

$$\begin{aligned}(K \ominus F) + A &= ((K \oplus F) \cap K) + A \\ &= ((K \oplus F) + A) \cap (K + A) \\ &= ((K \oplus F) + A) \cap K \\ &= K\end{aligned}$$

Suppose that $(\oplus 7)$ is satisfied. Note that $Cn(F_1) = Cn((F_1 \vee F_2) \cup F_1)$. By using $(\ominus 6)$ and $(\oplus 7)$, we have

$$\begin{aligned}K \ominus F_1 &= K \ominus ((F_1 \vee F_2) \cup F_1) &(2)\\ &\subseteq (K \oplus ((F_1 \vee F_2) \cup F_1))) \cap K &(3)\\ &\subseteq (K \oplus (F_1 \vee F_2) + F_1) \cap K &(4)\\ &\subseteq (K \oplus (F_1 \vee F_2) \cap K) + F_1 &(5)\\ &\subseteq K \ominus (F_1 \vee F_2) + F_1 &(6)\end{aligned}$$

Eq.(2) follows from $(\ominus 6)$. (3) follows by (Def $\ominus$). (4) follows by $(\ominus 7)$. (5) follows the fact:

$$(K_1 \cap K_2) + F = (K_1 + F) \cap (K_2 + F)$$

To show $(\ominus 8)$, assume that $F_1 \cup K \ominus (F_1 \vee F_2)$ is consistent, we need first to show that $F_1 \cup K \oplus (F_2 \vee F_2)$ is consistent. If not so, then there exists a finite subset $\bar{F}_1$ of F_1, so as to $K \oplus (F_1 \vee F_2) \vdash \neg(\wedge \bar{F}_1)$. It follows by Lemma 4.1 and (Def $\ominus$) that $K \ominus (F_1 \vee F_2) + (F_1 \vee F_2) \vdash \neg(\wedge \bar{F}_1)$, and then there are $A_1, \cdots, A_n \in F_1$ and $B_1, \cdots, B_n \in F_2$ such that $K \ominus (F_1 \vee F_2) \cup \{A_1 \vee B_1, \cdots, A_n \vee B_n\} \vdash \neg(\wedge \bar{F}_1)$. Hence $K \ominus (F_1 \vee F_2) \cup F_1 \vdash \neg(\wedge \bar{F}_1)$, so $K \ominus (F_1 \vee F_2) \cup F_1$ is inconsistent. But this contradicts the assumption. Therefore we have $F_1 \cup K \oplus (F_1 \vee F_2)$ is consistent. By using the identity $Cn(F_1) = Cn((F_1 \vee F_2) \cup F_1)$, $(\oplus 6)$ and $(\oplus 8)$, we obtain that

$$\begin{aligned}K \oplus (F_1 \vee F_2) + F_1 &\subseteq K \oplus ((F_1 \vee F_2) \cup F_1) \\ &= K \oplus F_1\end{aligned}$$

Then we have

$$\begin{aligned}K \ominus (F_1 \vee F_2) &\subseteq (K \oplus (F_1 \vee F_2)) \cap K \\ &\subseteq (K \oplus (F_1 \vee F_2) + F_1) \cap K \\ &\subseteq (K \oplus F_1) \cap K \\ &= K \ominus F_1\end{aligned}$$

as desired. □

5 Nice-Ordering Partition and Nice-Well-Ordering Partition

Epistemic entrenchment, introduced by Gärdenfors and Makinson (1988), plays a very important role in AGM theory, which is used to determine what is abandoned from a belief set and what is retained, when a contraction or a revision is carried out. Unfortunately, this notion has received several criticisms from different authors (see [21,25]). The main point is its intuitive inadequacy. In fact, there are two factors which determine the fate of a sentence in a belief set. One is the degree in which the epistemic agent believes it, and the other is its logical position in the belief set. The former proceeds from the consideration of the criterion of *informational economy* and the latter comes from the consideration of logical inference. In this section, we will try to develop several concepts specifying these two factors.

Every epistemic agent must have an estimate of his knowledge, but it is likely unrealistic to require him to arrange all his knowledge into a total ordering. For example, for the following believes:

$A =$'Germany is located in Europe'

$B =$'China is located in Asia'

It seems difficult to judge which should be believed more. It may, however, be possible to divide our knowledge base into groups, and then arrange these groups into a complete ordering according to the degrees in which we believe in them. This kind of arrangement is only dependent on our subjective sense. Maybe it is quite rough. In fact it need not be very 'accurate' because we will still adjust it further when necessary.

Definition 5.1. *Let Γ be a set of sentences, $\mathcal{P}$ be a partition of Γ* [3], *$<$ be a total ordering relation on $\mathcal{P}$. The triple $\Sigma = (\Gamma, \mathcal{P}, <)$ is called a total-ordering partition (TOP) of Γ.*

Note that Γ here need not to be closed. It can be conceived as a base of some belief sets. We will develop an approach to lift up a TOP of a belief set from that of its base.

For any $p \in \mathcal{P}$, if $A \in p$, then p is called the *rank* of A, denoted by $b(A)$. For technical consideration, we did not arrange the rank directly according to the degree of belief. Instead of that, the converse ordering of rank is the ordering the belief degree, that is, the higher the belief degree of a sentence is, the less rank of the sentence will be.

When $\Delta \subseteq \Gamma$, the following notation will be useful:

$$\Delta_p \overset{\text{Def}}{=} \Delta \cap p \tag{7}$$

$$\Delta_{<p} \overset{\text{Def}}{=} \bigcup_{q<p} \Delta_q \tag{8}$$

$$\Delta_{\leq p} \overset{\text{Def}}{=} \bigcup_{q\leq p} \Delta_q \tag{9}$$

Let Γ and F be sets of sentences. We define $\Gamma \perp F$ as the system of all maximal subsets of Γ that is consistent with F, that is,

[3] A partition of a set Γ is a disjoint family $\mathcal{P}$ of subsets of Γ such that

$$\Gamma = \bigcup\{p : p \in \mathcal{P}\}$$

where p needn't be nonempty.

$$\Gamma \perp F \overset{Def}{=} \{\Delta \subseteq \Gamma | \Delta \cup F \text{ is consistent and } \forall \Delta' \subseteq \Gamma(\Delta \subset \Delta' \to \Delta' \cup F \text{ is inconsistent}\}$$

It is easy to see that notation $K \perp F$ is a generalization of $K \perp A$ in the AGM framework, i.e., $K \perp \{\neg A\} = K \perp A$.

Observation 5.1. *If $\Sigma = (K, \mathcal{P}, <)$ is a total ordering partition of belief set K and F an arbitrary set of sentences, then for any $\Delta \in K \perp F$, Δ is closed.*

Proof. Straightforward from the construction of $K \perp F$. □

Ddefinition 5.2. *A total-ordering partition, $\Sigma = (\Gamma, \mathcal{P}, <)$, of Γ is called a nice-ordering partition (NOP) if it satisfies the following Logical Constraint:*
(L) If $A_1, \cdots, A_n \vdash B$, then $\sup\{b(A_1), \cdots, b(A_n)\} \geq b(B)$.

Definition 5.3. *Let $\Sigma = (\Gamma, \mathcal{P}, <)$ be a total ordering partition of Γ, F an arbitrary set of sentences, $\Gamma \Downarrow F$ is defined as the family of all subsets ,$\Delta = \bigcup_{p \in \mathcal{P}} \Delta_p$, of Γ, where for any $p \in \mathcal{P}$,*

$$\Delta_p \text{ is a maximal subset of } p \text{ such that } \Delta_{\leq p} \cup F \text{ is consistent}$$

The notation $\Downarrow$ comes from (Nebel 92 [21]).

Observation 5.2.

$$\Gamma \Downarrow F \subseteq \Gamma \perp F$$

In particular, if $\mathcal{P} = \{\Gamma\}$, then $\Gamma \Downarrow F = \Gamma \perp F$.

Proof. Suppose that $\Delta \in \Gamma \Downarrow F$ and $A \in \Gamma$ where $A \notin \Delta$. It is obvious that $\Delta \cup F$ is consistent. Let $A \in p (p \in \mathcal{P})$. By means of definition of Δ_p, we know that $\Delta_{\leq p} \cup \{A\} \cup F$ is inconsistent, so is $\Delta \cup \{A\} \cup F$. Hence $\Delta \in \Gamma \perp F$. □

Observation 5.3. *Let $\Sigma = (K, \mathcal{P}, <)$ be a nice-ordering partition of a belief set K and $\Delta \subseteq K$. Then we have*

$$\Delta \text{ is closed} \quad \text{if and only if} \quad \forall p \in \mathcal{P}(\Delta_{\leq p} \text{ is closed})$$

Proof. Suppose that Δ is closed. For any $p \in \mathcal{P}$, if $\Delta_{\leq p} \vdash B$, then there are $A_1, \cdots, A_n \in \Delta_{\leq p}$ so as to $A_1, \cdots, A_n \vdash B$. By definition of NOP we have that $\sup\{b(A_1), \cdots, b(A_n)\} \geq b(B)$, so $b(B) \leq p$, which means that $B \in K_{\leq p}$. Since Δ is closed, $B \in \Delta$, or $B \in \Delta_{\leq p}$.

Conversely, assume that for any $p \in \mathcal{P}$, $\Delta_{\leq p}$ is closed. If $\Delta \vdash B$, then there exist $A_1, \cdots, A_n \in \Delta$ such that $A_1, \cdots, A_n \vdash B$. Let $p = \sup\{b(A_1), \cdots, b(A_n)\}$. It follows that $A_1, \cdots, A_n \in \Delta_{\leq p}$, so $B \in \Delta_{\leq p}$, i.e. $B \in \Delta$ in terms of our assumption. □

Definition 5.4. *A nice-ordering partition $\Sigma = (\Gamma, \mathcal{P}, <)$ of Γ is called a nice-well-ordering partition (NWOP) of Γ if $<$ is a well-ordering relation on $\mathcal{P}$.*

A nature question, how we can arrive at such an NWOP of a belief set, arises immediately. The following lemma will tell us how to construct an NWOP of a belief set from an NWOP of its base.

Lemma 5.1. *Let D be a set of sentences and $K = Cn(D)$. If D has an* NWOP *$\Sigma^D = (D, \mathcal{P}^D, <^D)$, then there exists a* NWOP *$\Sigma^K = (K, \mathcal{P}^K, <^K)$, and vice versa, such that for any $A \in D$,*

$$b^D(A) = b^K(A) \tag{10}$$

We will call Σ^K the NWOP of K induced by Σ^D.

Proof. Let α be the order-type of $\mathcal{P}^D$ under well-ordering $<^D$ (see [17]). For any $\beta < \alpha$, we denote p^D_β as the βth element of $\mathcal{P}^D$ and $p^D_{\leq\beta} = \bigcup_{\gamma\leq\beta} p^D_\beta$. For any $\beta < \alpha$, we define p^K_β by recursion on β as follows:

$p^K_0 = Cn(p^D_0)$
$p^K_\beta = Cn(p^D_{\leq\beta})\backslash p^K_{<\beta}$, if $\beta < \alpha$

Let $\mathcal{P}^K = \{p^K_\beta | \beta < \alpha\}$. It is obvious that $K = \bigcup_{\beta<\alpha} p^K_\beta$ and $\mathcal{P}^K$ is a partition of K. Definite that

$$p^K_\gamma <^K p^K_\beta \qquad \text{iff} \qquad \gamma < \beta$$

To show Eq.(10), let $A \in D$ and $b^D(A) = \beta$. It is obvious that $b^K(A) \leq \beta$ by the construction of $\mathcal{P}^K$. If $b^K(A) = \gamma < \beta$, $A \in p^K_\gamma \subseteq Cn(p^D_{\leq\gamma})$, so, $A \in p^D_{\leq\gamma}$ because Σ^D is an NOP. We have got a contradiction. Thus $b^K(A) = b^D(A)$.

We now show that $\Sigma^K = (K, \mathcal{P}^K, <^K)$ is an NWOP. It is sufficient to show that Σ satisfies *logic constraint.* To do this, assume that $A \in K$. Then there is an ordinal β such that $A \in p^K_\beta$, i.e., $A \in Cn(p^D_{\leq\beta})$. Assume that $A_1, \cdots, A_n \in K$ such that $A_1, \cdots, A_n \vdash A$. If $\sup\{b^D(A_1), \cdots, b^D(A_n)\} < \beta$, $A \in Cn(p^D_{<\beta})$, which is contrary to $A \in p^K_\beta$. So $\sup\{b^D(A_1), \cdots, b^D(A_n)\} \geq \beta$. By Eq.(10) we have that

$$\sup\{b^K(A_1), \cdots, b^K(A_n)\} \geq \beta,$$

that is,

$$\sup\{b^K(A_1), \cdots, b^K(A_n)\} \geq b^K(A),$$

as desired.

For the other direction of the Lemma, suppose that $\Sigma = (K, P^K, <^K)$ is an NWOP of K. Let

$$\mathcal{P}^D = \{p^D | p^D = p^K \cap D\}$$

$$p^D <^D q^D \qquad \text{iff} \qquad p^K <^K q^K$$

It is not difficult to show that $\Sigma^D = (D, \mathcal{P}^D, <^D)$ is an NWOP of D. □

It is now clear that how we can arrive at an NWOP from any epistemic status. We can split the process into three stages. For any given knowledge database or belief base D, the first step is grouping D into several parts and arranging them in well-ordering in accordance with our belief degrees of sentences in each part. The second step is rearranging the partition above so that it satisfies *Logical Constraint*, so an NWOP of D is obtained. The final step is constructing an NWOP of belief set $Cn(D)$ from the NWOP of D by using the way provided by Lemma 5.1.

6 Relation of Epistemic Entrenchment and Nice-Order Partition

The main purpose of this section is to show the connections between the epistemic entrenchment ordering and the NOP structure.

Gärdenfors and Mankinson (1988) presented five postulates characterizing the notion of epistemic entrenchment:
(EE1) If $A \leq B$ and $B \leq C$, then $A \leq C$ (transitivity).
(EE2) If $A \vdash B$, then $A \leq B$ (dominance).
(EE3) For any A and B, $A \leq A \wedge B$ or $B \leq A \wedge B$ (conjunctiveness).
(EE4) When $K \neq K_{\perp}$, $A \notin K$ iff $A \leq B$, for all B (minimality).
(EE5) If $B \leq A$ for all B, then $\vdash A$ (maximality).

Postulate (EE1) says that $\leq$ is an ordering relation. (EE2) implies that $\leq$ is reflexive. And (EE1)–(EE3) imply that $\leq$ is connective. Thus $\leq$ is a complete preorder over L, particularly, over K. By using the way of equivalence partition, we can obtain a total-ordering partition over K. On the other hand, postulates (EE2) and (EE3) give epistemic entrenchment some *logical constraint* which are very similar to the *Logical Constraint* (L) of the NOP structure. More formally we have the following results.

Definition 6.1. *Let $\Sigma = (K, \mathcal{P}, <)$ be a nice-ordering partition of a belief set K, $K \neq K_{\perp}$, we define an order relation on L as follows:*
i) *if $B \in K$, then* $A \preceq B$ *if and only if* $b(B) \leq b(A)$ or $A \notin K$
ii) *If $B \notin K$, then* $A \preceq B$ *if and only if* $A \notin K$

Theorem 6.1. $\preceq$ *satisfies* (EE1)–(EE4).

Proof. (EE1) and (EE4) hold obviously.

To show (EE2) and (EE3), suppose that $A, B \in K$, other cases are trivial.

Assume that $A \vdash B$. Then $b(B) \leq sup\{b(A)\} = b(A)$ by using (L), and hence, $A \preceq B$, so (EE2) hold.

Since $A, B \vdash A \wedge B$, $b(A \wedge B) \leq sup\{b(A), b(B)\}$. On the other hand, $A \wedge B \vdash A$ and $A \wedge B \vdash B$, so $b(A) \leq b(A \wedge B)$ and $b(B) \leq b(A \wedge B)$. Therefore $b(A) = b(A \wedge B)$ or $b(B) = b(A \wedge B)$, this yields (EE3). □

Theorem 6.2. *If* $\preceq$ *satisfies* (EE1)–(EE3), *then there exists a nice-ordering partition $\Sigma = (K, \mathcal{P}, <)$ of K such that*

$$A \preceq B \quad \textit{iff} \quad b(B) \leq b(A)$$

Proof. Because $\preceq$ is reflexive, transitive and connective, it is a complete preorder. Define a equivalence relation on L as follows:

$$A \sim B \quad \text{if and only if} \quad A \preceq B \text{ and } B \preceq A$$

Then the quotient set $L/\sim$ makes a partition P over L. We use $\bar{A}$ to denote the corresponding equivalence class which include A. Let $\mathcal{P} = \{\bar{A} \cap K : A \in K\}$. For any $A, B \in K$, let

$$\bar{A} < \bar{B} \quad \text{if and only if} \quad A \not\preceq A \text{ and } B \preceq A$$

It is trivial to verify that $\Sigma = (K, \mathcal{P}, <)$ is a total-ordering partition on K.

In order to prove that Σ satisfies (L), assume that $A_1, \cdots, A_n \vdash B$. Then $A_1 \wedge \cdots \wedge A_n \vdash B$. By (EE2), $A_1 \wedge \cdots \wedge A_n \preceq B$. By repeated application of (EE3), we obtain

$$inf\{A_1, \cdots, A_n\} \preceq A_1 \wedge \cdots \wedge A_n \preceq B$$

and hence, $\sup\{b(A_1), \cdots, b(A_2)\} \geq b(B)$. □

The ordering characterized by (EE1)–(EE3) has received extensive investigations by several researchers (see [7, 12, 6]). In [12] Gärdenfors and Makinson (1994) gave this ordering another interpretation, namely *expectation*. They showed that 'nonmonotomic inferences could elegantly interpreted in terms of underlying expectations' and 'one could give a unified treatment of the theory of belief revision and that of nonmonotonic inference relation'. However, a question remained unknown, as its authors raised, where does the ordering or the expectation come from? Now if we employ the structure of nice-ordering partitions instead of the ordering of epistemic entrenchment, the problem may be readily solved.

7 Contraction Generated by Nice-Ordering Partition

In this section we will employ the notion of nice-ordering partition characterizing general revision and contraction operations. We will construct a contraction operator which satisfies all of the postulates for the general contraction. The corresponding revision operator is ready to be given by using (Def $\oplus$).

Definition 7.1. *Let $\Sigma = (K, \mathcal{P}, <)$ be an NOP on a belief set K (see Definition 5.2) and F be an arbitrary set of sentences. We define $K\dot{\ominus}F$, called NOP contraction, as follows:*

i) *If $F \cup K$ is consistent , then $K\dot{\ominus}F = K$.*

ii) *If $F \cup K$ is inconsistent, then $B \in K\dot{\ominus}F$ if and only if $B \in K$ and*

$$\exists A \in K(F \vdash \neg A \wedge \forall C \in K(A \vdash C \wedge F \vdash \neg C \rightarrow (b(A \vee B) < b(C) \vee \vdash A \vee B))) \quad (11)$$

In particular, when $F = \{\neg A\}$, then

$$B \in K\dot{\ominus}F \quad \text{iff} \quad B \in K \quad \text{and} \quad b(A \vee B) < b(A) \vee \vdash A \vee B \quad (12)$$

Theorem 7.1. *The function $\dot{\ominus}$ satisfies $(\ominus1)$–$(\ominus8)$.*

Proof. $(\ominus1)$. If $K \cup F$ is consistent, $(\ominus1)$ hold trivially. We may suppose that $K \cup F$ is inconsistent. Let $K\dot{\ominus}F \vdash B$. Then there are $B_1, \cdots, B_n \in K\dot{\ominus}F$ such that $\bigwedge_{i=1}^{n} B_i \vdash B$. By the definition of $\dot{\ominus}$, for any $i \leq n$, $B_i \in K$ and

$$\exists A_i \in K(F \vdash \neg A_i \wedge \forall C \in K(A_i \vdash C \wedge F \vdash \neg C \rightarrow (b(A_i \vee B_i) < b(C) \vee \vdash A_i \vee B_i)) \quad (13)$$

First we have that $\forall i \leq n(B_i \in K)$, so $B \in K$. And then let $A = \bigvee_{i=1}^{n} A_i$. So $F \vdash \neg A$. For any $C \in K$, if $A \vdash C$ and $F \vdash \neg C$, then $\forall i \leq n(A_i \vdash C)$. It follows immediately from (13) that $\forall i \leq n(b(A_i \vee B_i) < b(C) \vee \vdash A_i \vee B_i)$. If $\forall i \leq n(\vdash A_i \vee B_i)$, then $\forall i \leq n(\vdash A \vee B_i)$, so $\vdash A \vee B$. Otherwise, let $b(A_{i_0} \vee B_{i_0}) = \sup_{1 \leq i \leq n} \{b(A_i \vee B_i)\}$ and $\nvdash A_{i_0} \vee B_{i_0}$, hence,

$$\begin{aligned} b(A_{i_0} \vee B_{i_0}) &= \sup_{1 \leq i \leq n} \{b(A_i \vee B_i)\} \geq b(\bigwedge_{i=1}^{n} (A_i \vee B_i)) \\ &\geq b(\bigwedge_{i=1}^{n} (A \vee B_i)) = b(A \vee \bigwedge_{i=1}^{n} B_i) \\ &= b(A \vee B). \end{aligned}$$

Since $\not\vdash A_{i_0} \vee B_{i_0}$, we obtain that $b(A \vee B) < b(C)$.

($\ominus$2) and ($\ominus$3) are trivial.

($\ominus$4). Suppose that F is consistent. We are going to show that $F \cup K\dot{\ominus}F$ is consistent. If $F \cup K$ is consistent, it holds trivially. We may thus assume that $F \cup K$ is inconsistent. If $F \cup (K\dot{\ominus}F)$ is inconsistent. Then there exists a finite subset $\bar{F}$ of F such that $K\dot{\ominus}F \vdash \neg(\wedge\bar{F})$, that is, $\neg(\wedge\bar{F}) \in K\dot{\ominus}F$. Hence, by the definition of $\dot{\ominus}$, there exists $A_0 \in K$ such that $F \vdash \neg A_0$ and

$$\forall C \in K(A_0 \vdash C \wedge F \vdash \neg C \to (b(A_0 \vee \neg(\wedge\bar{F})) < b(C) \vee \vdash A_0 \vee \neg(\wedge\bar{F}))) \tag{14}$$

Since $A_0 \vdash A_0 \vee \neg(\wedge\bar{F})$ and $F \vdash \neg(A_0 \vee \neg(\wedge\bar{F}))$, by using (14), $b(A_0 \vee \neg(\wedge\bar{F})) < b(A_0 \vee \neg(\wedge\bar{F})) \vee \vdash A_0 \vee \neg(\wedge\bar{F})$. We conclude that $\vdash A_0 \vee \neg(\wedge\bar{F})$, i.e., $\neg A_0 \vdash \neg(\wedge\bar{F})$, and hence, $F \vdash \neg(\wedge\bar{F})$. That means that F is inconsistent, which contradicts the supposition. Thus when F is consistent, so is $F \cup (K\dot{\ominus}F)$. On the other hand, suppose that F is inconsistent. Then for any sentence A, $F \vdash \neg A$, even though A is a tautology. Just assume that A is a tautology. For any $B \in K$ and any $C \in K$, we always have $\vdash A \vee B$. Thus $B \in K\dot{\ominus}F$ in terms of the definition of $\dot{\ominus}$.

($\ominus$5). For any $A \in K$, if $F \vdash \neg A$, then $F \cup K$ is inconsistent. We are going to show that $K = K\dot{\ominus}F + A$. For any $B \in K$, because $\vdash A \vee (A \to B)$, so $A \to B \in K\dot{\ominus}F$ according to (11). We arrive at $B \in K\dot{\ominus}F + A$.

($\ominus$6) is trivial.

($\ominus$7). If $F_1 \cup K$ or $(F_1 \vee F_2) \cup K$ is consistent, then ($\ominus$7) holds obviously. So we suppose that both $F_1 \cup K$ and $(F_1 \vee F_2) \cup K$ are inconsistent. Let $B \in K\dot{\ominus}F_1$. According to the definition of $\dot{\ominus}$, we have

$$\exists A_0 \in K(F_1 \vdash \neg A_0 \wedge \forall C \in K(A_0 \vdash C \wedge F_1 \vdash \neg C \to (b(A_0 \vee B) < b(C) \vee \vdash A_0 \vee B))) \tag{15}$$

On the other hand, since $(F_1 \vee F_2) \cup K$ is inconsistent, so there exists $A \in K$ such that $F_1 \vee F_2 \vdash \neg A$, so $F_1 \vdash \neg A$. Therefore, there exists a finite subset $\bar{F}_1$ of F_1 such that $\bar{F}_1 \vdash \neg A_0 \wedge \neg A$, hence $A_0 \vdash \neg(\wedge\bar{F}_1)$. For any $C \in K$, if $A \vdash C$ and $F_1 \vee F_2 \vdash \neg C$, then $F_1 \vdash (\wedge\bar{F}_1) \wedge \neg C$ and $A_0 \vdash \neg(\wedge\bar{F}_1) \vee C$. By using (15), we have

$$(b(A_0 \vee B) < b(\neg(\wedge\bar{F}_1) \vee C)) \vee (\vdash A_0 \vee B) \tag{16}$$

If $\vdash A_0 \vee B$, by $A_0 \vdash \neg(\wedge\bar{F}_1)$, we have $\vdash \neg(\wedge\bar{F}_1) \vee B$, i.e., $\vdash A \vee \neg(\wedge\bar{F}_1) \vee B$. If $b(A_0 \vee B) < b(\neg(\wedge\bar{F}_1) \vee C)$, noting that $\bar{F}_1 \vdash \neg A$ imply $\neg(\wedge\bar{F}_1) \dashv\vdash \neg(\wedge\bar{F}_1) \vee A$, we have

$$\begin{aligned} b(A \vee \neg(\wedge\bar{F}_1) \vee B) &= b(\neg(\wedge\bar{F}_1) \vee B) \\ &\leq b(A_0 \vee B) \\ &< b(\neg(\wedge\bar{F}_1) \vee C) \\ &\leq b(C) \end{aligned}$$

that is, $b(A \vee \neg(\wedge\bar{F}_1) \vee B) \leq b(C)$. So we conclude

$$(b(A \vee \neg(\wedge\bar{F}_1) \vee B) < b(C)) \vee (\vdash A \vee \neg(\wedge\bar{F}_1) \vee B) \tag{17}$$

By means of the definition of $\dot{\ominus}$, we have $\neg(\wedge\bar{F}_1) \vee B \in K\dot{\ominus}(F_1 \vee F_2)$, so $B \in (K\dot{\ominus}(F_1 \vee F_2) + F_1) \cap K$.

($\ominus$8). Suppose that $F_1 \cup K\dot{\ominus}(F_1 \vee F_2)$ is consistent. we are going to show that $K\dot{\ominus}(F_1 \vee F_2) \subseteq K\dot{\ominus}F_1$.

If $F_1 \cup K$ or $(F_1 \vee F_2) \cup K$ is consistent, then ($\ominus$8) holds obviously. We may thus suppose that $F_1 \cup K$ and $(F_1 \vee F_2) \cup K$ are inconsistent. Assume that $B \in K\dot{\ominus}(F_1 \vee F_2)$. By the definition of $\dot{\ominus}$, there exists $A_0 \in K$ such that $F_1 \vee F_2 \vdash \neg A_0$ and

$$\forall C \in K(A_0 \vdash C \wedge F_1 \vee F_2 \vdash \neg C \rightarrow (b(A_0 \vee B) < b(C) \vee \vdash A_0 \vee B)) \tag{18}$$

We need to show $B \in K\dot{\ominus}F_1$. Suppose to the contrary that $B \notin K\dot{\ominus}F_1$. Again by the definition of $\dot{\ominus}$, we have

$$\forall A \in K(F_1 \vdash \neg A \rightarrow \exists C \in K(A \vdash C \wedge F_1 \vdash \neg C \wedge b(C) \leq b(A \vee B) \wedge \nvdash A \vee B))$$

In particular, since $F_1 \vdash \neg A_0$ there is $C_0 \in K$ such that

$$A_0 \vdash C_0 \wedge F_1 \vdash \neg C_0 \wedge b(C_0) \leq b(A_0 \vee B) \wedge \nvdash A_0 \vee B \tag{19}$$

Let $\bar{F}_1$ be a finite subset of F_1 so as to $\bar{F}_1 \vdash \neg A_0 \wedge \neg C_0$. Because $F_1 \cup K\dot{\ominus}(F_1 \vee F_2)$ is consistent, $\neg(\wedge\bar{F}_1) \notin K\dot{\ominus}(F_1 \vee F_2)$, so by $F_1 \vee F_2 \vdash \neg A_0$, there exists $C_1 \in K$ such that

$$A_0 \vdash C_1 \wedge (F_1 \vee F_2) \vdash \neg C_1 \wedge b(C_1) \leq b(A_0 \vee \neg(\wedge\bar{F}_1)) \wedge \nvdash A_0 \vee \neg(\wedge\bar{F}_1) \tag{20}$$

Let $C = C_0 \wedge C_1$, then $A_0 \vdash C_0 \wedge C_1$ and $F_1 \vee F_2 \vdash \neg C_0 \vee \neg C_1$. Applying (18) to C we get

$$b(A_0 \vee B) < b(C_0 \wedge C_1) \vee \vdash A_0 \vee B \tag{21}$$

By (19) we have $\nvdash A_0 \vee B$, so $b(A_0 \vee B) < b(C_0 \wedge C_1)$. On the other hand, we have

$$\begin{aligned} b(C_1) &\leq b(A_0 \vee \neg(\wedge\bar{F}_1)) && (22)\\ &\leq b(\neg(\wedge\bar{F}_1)) && (23)\\ &\leq b(C_0) && (24) \end{aligned}$$

Inequality (22) follows from (20). (23) follows from the properties of NOP. (24) follows from $\bar{F}_1 \vdash \neg C_0$. Thus $b(C_1) \leq b(C_0)$, so $b(C_0 \wedge C_1) = b(C_0)$.

By putting (21) and (19) together with the above result we obtain that $b(A_0 \vee B) < b(C_0) \leq b(A_0 \vee B)$, which is impossible. This contradiction concludes the proof. □

8 Contraction Generated by Nice-Well-Ordering Partition

A very important problem in studies of Belief Revision is computation of revision and contraction operations. The approach based on nice-ordering partition or epistemic entrenchment is suitable for modeling these two operations, but not for computing them, even though both K and F are finite sets. Nebel (1992)[21] introduced a constructive approach to specify his base revision operation. In this section, we will use his method for reference to attack the problem of computation in the general framework of belief revision.

Definition 8.1. *Let* $\Sigma = (K, \mathcal{P}, <)$ *be an NWOP of a belief set* K *(see Definition* 5.4) *and* F *an arbitrary set of sentences. We define* $K\hat{\ominus}F$*, called NWOP contraction, as follows:*
i) *If* F *is inconsistent, then*

$$K\hat{\ominus}F \overset{Def}{=} K$$

ii) *If* F *is consistent, then*

$$K\hat{\ominus}F \overset{Def}{=} \bigcap K \Downarrow F$$

Theorem 8.1. *Let* $\Sigma = (K, \mathcal{P}, <)$ *be an* NWOP *of* K *and* F *an arbitrary set of sentences. Then*

$$K\dot{\ominus}F = K\hat{\ominus}F \tag{25}$$

Proof. '$\subseteq$'. It is trivial for the case when F is inconsistent or $K \cup F$ is consistent. We may suppose that F is consistent and $K \cup F$ is inconsistent. Let $B \in K\dot{\ominus}F$. It follows that $B \in K$ and there exists $A \in K$ such that $F \vdash \neg A$ and

$$\forall C \in K(A \vdash C \wedge F \vdash \neg C \to (b(A \vee B) < b(C) \vee \vdash A \vee B)) \tag{26}$$

We need to show that $B \in K\hat{\ominus}F$. Assume that $B \notin K\hat{\ominus}F$, that is, there is $\Delta \in K \Downarrow F$ such that $B \notin \Delta$. Because $\Delta \in K \perp F$, $\neg A \vee B \in \Delta$, or $A \vee B \notin \Delta$. Let $b(A \vee B) = q$. It follows that $\Delta_{\leq q} \cup F \cup \{A \vee B\}$ is inconsistent. So there exists a finite subset $\bar{F}$ of F such that $\bar{F} \vdash \neg A$ and $\Delta_{\leq q} \cup \{A \vee B\} \vdash \neg(\wedge\bar{F})$. That means $\neg(\wedge\bar{F}) \in K$ and $b(\neg(\wedge\bar{F})) \leq q$. Let $C = A \vee \neg(\wedge\bar{F})$. Then $A \vdash C$ and $F \vdash \neg C$. By (26) we have

$$q = b(A \vee B) < b(A \vee \neg(\wedge\bar{F})) \leq b(\neg(\wedge\bar{F})) \text{ or } \vdash A \vee B$$

Both of them are impossible. Thus $B \in K\hat{\ominus}F$.

'$\supseteq$'. It is trivial for the case when F is inconsistent or $K \cup F$ is consistent. We suppose that F is consistent and $K \cup F$ is inconsistent. Let $B \in K\hat{\ominus}F$. We need to show that $B \in K\dot{\ominus}F$. Suppose that $B \notin K\dot{\ominus}F$. It follows that

$$\forall A \in K(F \vdash \neg A \to \exists C \in K(A \vdash C \wedge F \vdash \neg C \wedge b(C) \leq b(A \vee B) \wedge \nvdash A \vee B)) \tag{27}$$

Because $K \cup F$ is inconsistent and $<$ is a well-order over $\mathcal{P}$, there exists $p^0 (\in \mathcal{P})$ such that p^0 is the minimal element of set $\{b(A) : A \in K \wedge F \vdash \neg A\}$. Let $A_0 \in p^0$ where $F \vdash \neg A_0$. For any $\Delta \in K \Downarrow F$, by $B \in K\hat{\ominus}F$, we have that $B \in \Delta$, i.e., $A_0 \vee \neg B \notin \Delta$. Thus we obtain that $A_0 \vee \neg B \notin \bigcup K \Downarrow F$. Let $b(A_0 \vee \neg B) = p$. If there exists $\Delta \in K \Downarrow F$ such that $\Delta_{<p} \cup \{A_0 \vee \neg B\} \cup F$ is consistent, we can construct a set $\Delta' \in K \Downarrow F$ such that $\forall q < p(\Delta_q = \Delta'_q)$ and $A_0 \vee \neg B \in \Delta'$. But $A_0 \vee \neg B \in \Delta'$ contradicts to $A_0 \vee \neg B \notin \bigcup K \Downarrow F$. Therefore, for any $\Delta \in K \Downarrow F$, $\Delta_{<p} \cup \{A_0 \vee \neg B\} \cup F$ is inconsistent. It follows that there exists a finite subset $\bar{F} \subseteq F$ such that $\bar{F} \vdash \neg A_0$ and $\Delta_{<p} \cup \{A_0 \vee \neg B\} \vdash \neg(\wedge\bar{F})$. So $\neg(\wedge\bar{F}) \in K$ and $b(\neg(\wedge\bar{F})) \leq p$. On the other hand, because $\Delta_{<p} \vdash (\neg A_0 \wedge B) \vee \neg(\wedge\bar{F})$, $b((\neg A_0 \wedge B) \vee \neg(\wedge\bar{F})) < p$. We conclude that

$$\begin{aligned} b(B \vee \neg(\wedge\bar{F})) &= b((\neg A_0 \wedge B) \vee A_0 \vee \neg(\wedge\bar{F})) \\ &= b((\neg A_0 \wedge B) \vee \neg(\wedge\bar{F})) \\ &< p \end{aligned}$$

Let $A = \neg(\wedge\bar{F})$. Since $F \vdash \neg A$, by using (27), we obtain that there is $C_0 \in K$ such that

$$A \vdash C_0 \wedge F \vdash \neg C_0 \wedge b(C_0) \leq b(A \vee B) \wedge \nvdash A \vee B$$

Hence, $b(C_0) \leq b(A \vee B) = b(\neg(\wedge\bar{F}) \vee B) < p$. However, $p = b(A_0 \vee \neg B) \leq b(A_0) = p^0$, i.e., $p \leq p^0$, so $b(C_0) < p^0$, which contradict the construction of p^0. □

9 Application to Default Logic

In [21], Nebel extended Pool's *theory formation approach* and Brewka's *level default theories* to *ranked default theories* (RDT). A RDT Δ is a pair $(\mathcal{D}, \mathcal{F})$, where $\mathcal{D}$ is a finite family of finite sets of sentences interpreted as ranked defaults and $\mathcal{F}$ is finite set of sentences interpreted as hard facts. Nebel showed that there exists a tight connection between belief revision and finite ranked default theories. We now try to generalize it to the general case.

A default theory $\Delta = (D, F)$ is called *nice-well-ordering parted default theory* (NWOP DT) if there exists an NWOP, $\Sigma = (D, \mathcal{P}, <)$, of D, where D is a set of defaults and F is a set of propositional sentences, interpreted as hard facts.

A set of sentences $E = Cn((\bigcup_{p \in \mathcal{P}} R_p) \cup F)$ is called an *extension* of Δ if for all $p \in \mathcal{P}$, R_p is the maximal subset of $Cn(D_{\leq p}) \backslash R_{<p}$ such that $(\bigcup_{q \leq p} R_q) \cup F$ is consistent[4].

Similar to [21] we call a sentence A *strongly provable* in Δ, denoted by $\Delta |\!\sim A$, if and only if for all extensions E of Δ, $A \in E$ (see [21] p.74).

Lemma 9.1.

$$\bigcap_{\Delta \in K \Downarrow F} (\Delta + F) = (\bigcap_{\Delta \in K \Downarrow F} \Delta) + F$$

Proof. The direction '$\supseteq$' is obvious. For the other direction suppose that $K \cup F$ is inconsistent. Then there exists a finite subset $\bar{F}$ of F such that $\neg(\wedge\bar{F}) \in K$. For any $A \in \bigcap_{\Delta \in K \Downarrow F} (\Delta + F)$, if $A \notin \bigcap_{\Delta \in K \Downarrow F} \Delta + F$, $\neg(\wedge\bar{F}) \vee A \notin \bigcap_{\Delta \in K \Downarrow F} \Delta$, i.e., there is $\Delta' \in K \Downarrow F$ such that $\neg(\wedge\bar{F}) \vee A \notin \Delta'$. Hence $\Delta' \cup \{\neg(\wedge\bar{F}) \vee A\} \cup F$ is inconsistent, i.e., $\Delta' \cup F \vdash \wedge\bar{F} \wedge \neg A$, or, $\Delta' \cup F \vdash \neg A$. On the other hand, by $A \in \bigcap_{\Delta \in K \Downarrow F} (\Delta + F)$, we have $A \in \Delta' + F$, so $\Delta' \cup F \vdash A$, which is in contradiction to consistency of Δ'. □

Theorem 9.1. *Let* $\Delta = (D, F)$ *be an* NWOP DT, *and* $\Sigma^D = (D, \mathcal{P}^D, F)$ *be an NWOP of D. Let* $K = Cn(D)$ *and* $\Sigma = (K, \mathcal{P}^K, <^K)$ *be the NWOP of K induced by* Σ^D. *Then for all sentence A in L,*

$$\Delta |\!\sim A \text{ iff } A \in K \hat{\oplus} F$$

where $K \hat{\oplus} F = K \hat{\ominus} F + F$.

Proof. It is sufficient by Lemmas 5.1 and 9.1 to show that

$$\{E | \quad E \text{ is an extension of } \Delta\} = \{K' + F | \quad K' \in K \Downarrow F\}$$

[4] Here the construction of R_p is different to Nebel's. R_p need not to be a subset of p.

which follows immediately from the constructions of extension of an NWOP DT and $K \Downarrow F$. □

Makinson and Gärdenfors in [19] showed that the postulates for belief revision can be translated into postulates for nonmonotonic logic. However, as they pointed out, the idea of infinite revision function would make better intuitive sense. In fact, if we follows their ways of translating $F |\!\sim A$ as $A \in K \oplus F$, then it is easy to show that the nonmonotonic inference relation $|\!\sim$ satisfies all of the Gabbay's postulates for nonmonotonic inference:

1. $F, A |\!\sim A$ (reflexivity)
2. If $F |\!\sim A$ and $F, A |\!\sim B$ then $F |\!\sim B$ (transitivity)
3. If $F |\!\sim A$ and $F |\!\sim B$, then $F, A |\!\sim B$ (restricted monotonicity)

We will discuss this topic in a separate paper.

10 Conclusion

We have extended AGM's system of belief revision to a more general framework, in which we are able to deal with the revision not only by single sentences but also by any sets of sentences, especially by infinite sets. We have introduced the notion of nice-ordering partitions to characterize belief revision, which seems to have a better intuitive sense and is more computational than epistemic entrenchment. A computational-oriented approach to deal with belief revision has been given. A general framework for belief revision will provide us with more room to study the relationship between belief revision and nonmonotonic logic as we done in the last section.

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